# Chapter 6. Applications of Definite Integrals

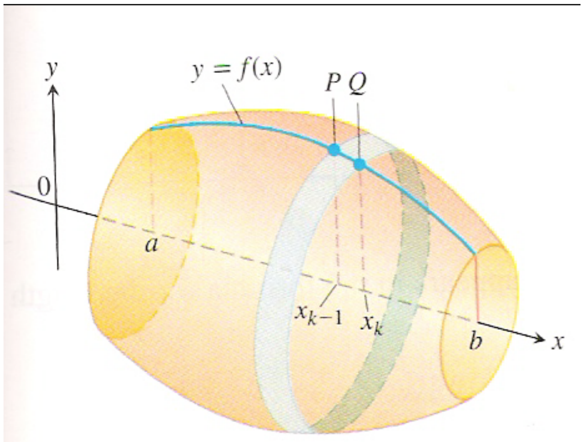
## 6.5. Areas of Surfaces of Revolution and the Theorems of Pappus

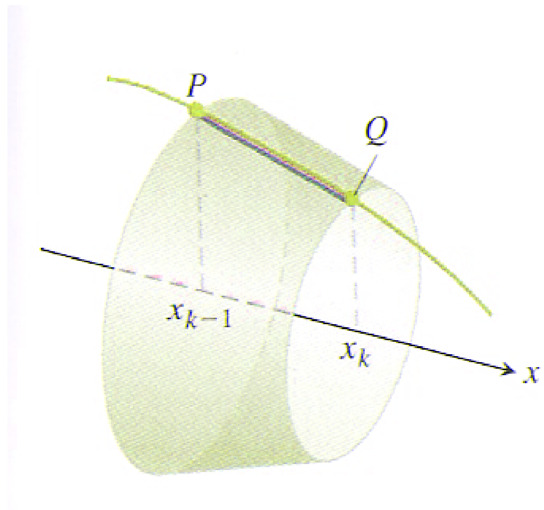
**Recall.** A differential of arclength $s$ of the function $y = f(x)$ is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

**Example.** Set up an integral for the length of $y = x^2$ and $x \in [0, 1]$. Notice that you cannot evaluate the resulting integral. This innocent looking integral will require several of the techniques developed in Chapter 8.

**Note.** In order to find the surface area that results from revolving an arc, we partition the arc into pieces and revolve them. If we follow the "heuristic" approach (as opposed to explicitly going through the partition, the slices, the norm of the partition, and limits), then a slice of arclength produces a ring of width $ds$ and radius $x$ (if the arc was revolved about the $y$-axis) or $y$ (if the arc was revolved about the $x$-axis). Therefore the area of the ring, which is a differential of the area of the surface, is $dS = \text{ radius } \times ds$.




Figures 6.44 and 6.45, page 437.

**Definition.** If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the *area* of the surface generated by revolving the curve $y = f(x)$ about the $x$-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

**Example.** Page 445 number 14.

**Definition.** If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the $y$-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2}\, dy.$$

**Definition.** If a smooth curve $x = f(t)$, $y = g(t)$, $t \in [a, b]$, is traversed exactly once as $t$ increases from $a$ to $b$, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows:

**1.** Revolution about the $x$-axis ($y \geq 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b 2\pi g(t) \sqrt{(f'(t))^2 + (g'(t))^2}\, dt.$$

**2.** Revolution about the $y$-axis ($x \geq 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b 2\pi f(t) \sqrt{(f'(t))^2 + (g'(t))^2}\, dt.$$

**Example.** Page 446 number 34.

**Theorem 1. Pappus's Theorem for Volumes**

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If $\rho$ is the distance from the axis of revolution to the centroid, then $V = 2\pi\rho A$.

**Example.** Page 443 Example 5.

**Theorem 2. Pappus's Theorem for Surface Areas**

If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length of the arc times the distance traveled by the arc's centroid during the revolution. If $\rho$ is the distance from the axis of revolution to the centroid, then $S = 2\pi\rho L$.

**Example.** Page 447 number 44a.