Chapter 7. Transcendental Functions

7.1. Inverse Functions and Their Derivatives

Definition. A function $f(x)$ is *one-to-one* on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

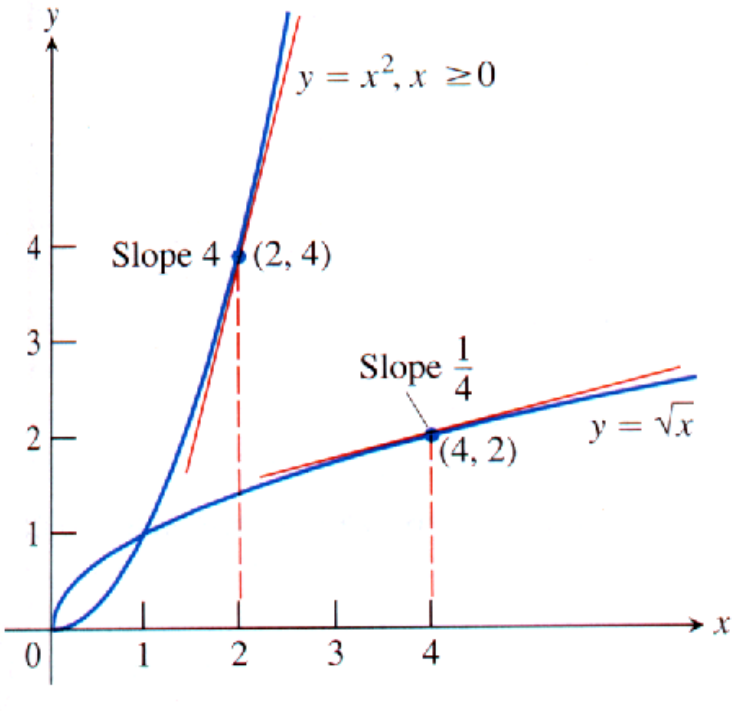
Note. A function $y = f(x)$ is one-to-one if and only if its graph interesects each horizontal line at most once.

Definition. Suppose that f is a one-to-one function on a domain D with range R. The *inverse function* f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

Note. Every one-to-one function (that is, a function for which any horizontal line intersects the graph of the function at most once) has an inverse. If the graph of $y = f(x)$ is reflected about the line $y = x$, then it will lie on top of the graph of $y = f^{-1}(x)$.



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Note. If we have a function $y = f(x)$, then we can find $f^{-1}(x)$ by (1) swapping x and y in the equation $y = f(x)$ and (2) solving for y.

Example. If $f(x) = 4x^2 + 2x - 5$ for $x \leq -1/4$ (notice the condition $x \leq -1/4$ is necessary to make the function one-to-one), then $f^{-1}(x) =$?

Example. Page 473 number 10.

Theorem 1. If f is differentiable at every point of an interval I and df/dx is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

Proof. By definition of inverse function, $f^{-1}(f(x)) = x$ for all $x \in I$. Differentiating this equation, we have by the Chain Rule:

$$\frac{d}{dx}\left[f^{-1}(f(x))\right] = \frac{d}{dx}[x]$$

$$f^{-1\prime}(f(x))f'(x) = 1$$

$$f^{-1\prime}(f(x)) = \frac{1}{f'(x)}.$$

Plugging in $x = f^{-1}(b)$, we get the theorem. *Q.E.D.*

Example. Page 474 number 30, 44, 46.