Chapter 7. Transcendental Functions

7.2. Natural Logarithms

Note. In this section, we introduce the natural logarithm function using definite integrals. However, to justify using the *logarithm* terminology we must show that the function we introduce satisfies the usual properties of logarithms. We will do so using our definition and the "calculus properties" which it satisfies.

Definition. For $x > 0$, define the *natural logarithm* function as

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

Note. It follows from the definition that for $x \geq 1$, $\ln x$ is the area under the curve $y = 1/t$ for $t \in [1, x]$. All we can currently tell from the definition, is that $\ln x < 0$ for $x \in (0, 1)$, $\ln 1 = 0$, and $\ln x > 0$ for $x \in (1, \infty)$. We also see that $\ln x$ is an INCreasing function of x.

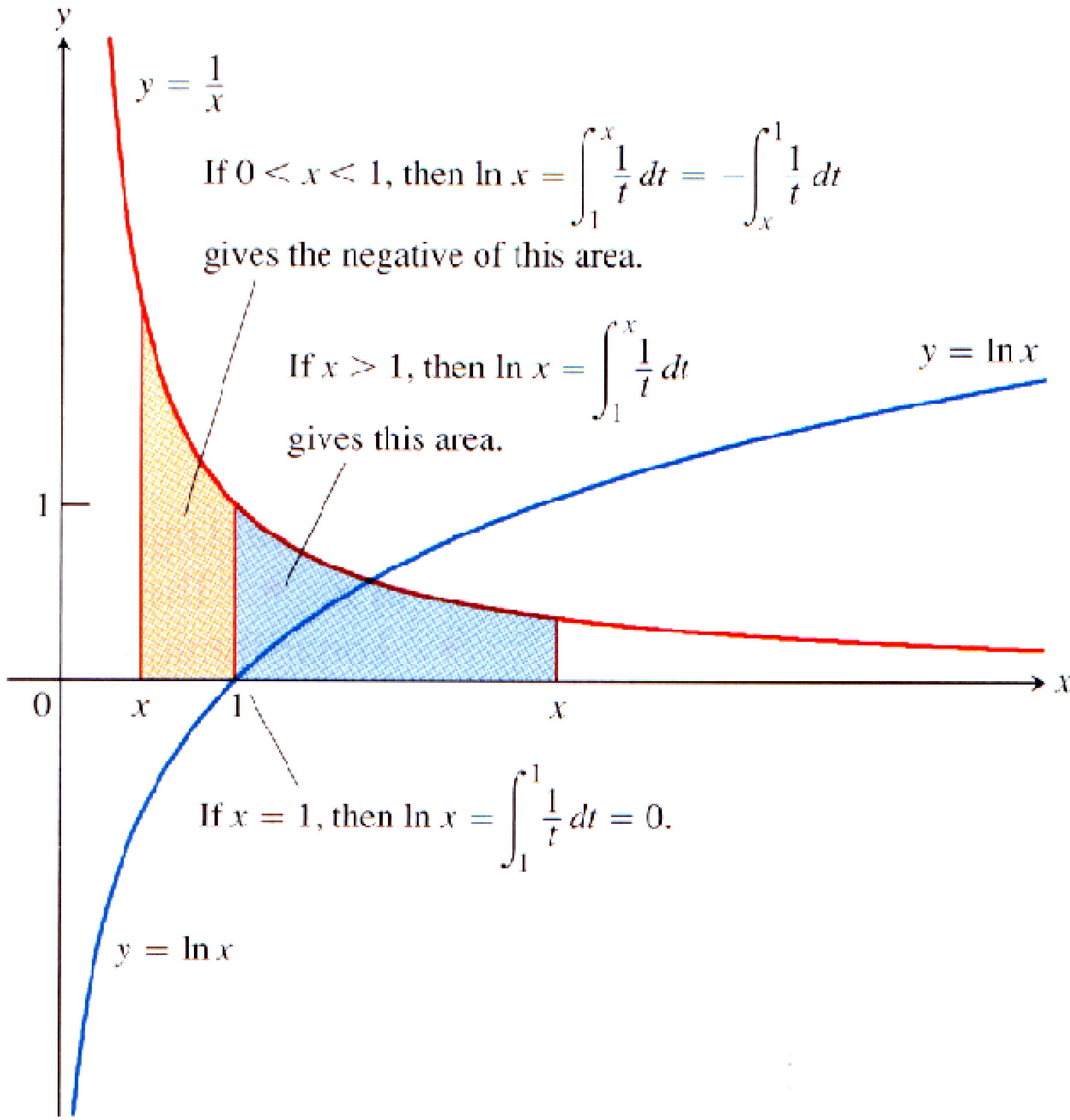

Figure 7.9 page 477

Definition. The number e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1.$$

Theorem. For $x > 0$ we have

$$\frac{d}{dx}[\ln x] = \frac{1}{x}.$$

If $u = u(x)$ is a differentiable function of x, then for all x such that $u(x) > 0$ we have

$$\frac{d}{dx}[\ln u] = \frac{d}{dx}[\ln u(x)] = \frac{1}{u}\left[\frac{du}{dx}\right] = \frac{1}{u(x)}[u'(x)].$$

Proof. We have b the Fundamental Theorem of Calculus Part 2:

$$\frac{d}{dx}[\ln x] = \frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] = \frac{1}{x}.$$

By the Chain Rule

$$\frac{d}{dx}[\ln u(x)] = \frac{d}{du}[\ln u]\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

Q.E.D.

Examples. Page 484 numbers 14 and 26.

Theorem 2. Properties of Logarithms. For any numbers $a > 0$ and $x > 0$ we have

1. $\ln ax = \ln a + \ln x$
2. $\ln \dfrac{a}{x} = \ln a - \ln x$
3. $\ln \dfrac{1}{x} = -\ln x$
4. $\ln x^r = r \ln x$ where r is rational.

Proof. First for **1**. Notice that

$$\frac{d}{dx}[\ln ax] = \frac{1}{ax}\frac{d}{dx}[ax] = \frac{1}{ax}a = \frac{1}{x}.$$

This is the same as the derivative of $\ln x$. Therefore by Corollary 1 to the Mean Value Theorem, $\ln ax$ and $\ln x$ differ by a constant, say $\ln ax = \ln x + k_1$ for some constant k_1. By setting $x = 1$ we need $\ln a = \ln 1 + k_1 = 0 + k_1 = k_1$. Therefore $k_1 = \ln a$ and we have the identity $\ln ax = \ln a + \ln x$.

Now for **2**. We know by **1**:

$$\ln \frac{1}{x} + \ln x = \ln\left(\frac{1}{x}x\right) = \ln 1 = 0.$$

Therefore $\ln \dfrac{1}{x} = -\ln x$. Again by **1** we have

$$\ln \frac{a}{x} = \ln\left(a\frac{1}{x}\right) = \ln a + \ln \frac{1}{x} = \ln a - \ln x.$$

Finally for **4**. We have by the Chain Rule (in the form of the previous theorem):

$$\frac{d}{dx}[\ln x^n] = \frac{1}{x^n}\frac{d}{dx}[x^n] = \frac{1}{x^n}nx^{n-1} = n\frac{1}{x} = n\frac{d}{dx}[\ln x] = \frac{d}{dx}[n\ln x].$$

As in the proof of **1**, since $\ln x^n$ and $n\ln x$ have the same derivative, we have $\ln x^n = n\ln x + k_2$ for some k_2. With $x = 1$ we see that $k_2 = 0$ and we have $\ln x^n = n\ln x$. *Q.E.D.*

Note. We can sometimes use the properties of this theorem to simplify expressions before differentiating them.

Example. Page 464 number 22. (Notice the use of the "square bracket notation.")

Note. We can, in fact, take the logarithm of a complicated function before differentiating it and then implicitly differentiate the result. This process is called *logarithmic differentiation*. It allows us to use the laws of logarithms instead of some of the complicated rules of differentiation.

Example. Page 485 number 66.

Theorem. If u is a differentiable function that is never zero then

$$\int \frac{1}{u}\,du = \ln|u| + C = \{\ln|u(x)| + k \mid k \in \mathbb{R}\}.$$

Proof. We know the result holds for $u(x) > 0$. We must only establish it for $u(x) < 0$. Notice that when $u(x) < 0$, $-u(x) > 0$, and $|u(x)| = -u(x)$

$$\int \frac{1}{u}\,du = \int \frac{1}{-u} d(-u) = \ln(-u) + C = \ln|u| + C.$$

Q.E.D.

Note. We can also express the previous theorem as

$$\int \frac{1}{u(x)} u'(x)\,dx = \ln|u(x)| + C.$$

where $u(x)$ is nonzero.

Examples. Page 484 numbers 42 and 44.

Theorem. For $u = u(x)$ a differentiable function,

$$\int \tan u\, du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u\, du = \ln|\sin u| + C = -\ln|\csc u| + C.$$

Proof. Both follow from u-substitution:

$$\begin{aligned}
\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx \\
& \text{Let } u = \cos x \\
& \text{then } du = -\sin x\, dx \\
&= \int \frac{-du}{u} = -\int \frac{du}{u} \\
&= -\ln|u| + C = -\ln|\cos x| + C \\
&= \ln \frac{1}{|\cos x|} + C \\
&= \ln|\sec x| + C.
\end{aligned}$$

Next

$$\begin{aligned}
\int \cot x\, dx &= \int \frac{\cos x}{\sin x}\, dx \\
& \text{Let } u = \sin x \\
& \text{then } du = \cos x\, dx \\
&= \int \frac{du}{u} = \int \ln|u| + C \\
&= \ln|\sin x| + C = -\ln \frac{1}{|\sin x|}
\end{aligned}$$

$$= -\ln|\csc x| + C.$$

The theorem follows by another application of u-substitution. *Q.E.D.*

Example. Page 484 number 52.

Example. Page 485 number 74.