Chapter 7. Transcendental Functions

7.3. The Exponential Functions

Note. On page 486 it is argued (not very rigorously) that $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to 0^+} \ln x = -\infty$. Therefore (since $\ln x$ is continuous — why's that?) the range of $\ln x$ is $(-\infty, \infty)$. We also know from the definition that $\ln x$ is a (strictly) increasing function. Therefore it is one-to-one and has an inverse.

Note. We have defined the number e as $e = \ln^{-1} 1$. We will express e as a limit at the end of this section. One finds numerically that $e \approx 2.718281828459045$.

Note. Notice that we can express e in terms of areas by writing

$$1 = \int_1^e \frac{1}{t}\, dt.$$

Note. The number e is an example of a *transcendental number* (as opposed to an *algebraic number*). The number π is also transcendental. The text mentions this on page 487. The six trigonometric functions, the logarithm functions, and exponential functions (to be defined soon) are transcendental (thus the title of this chapter).

Definition. Define the *natural exponential function* as $e^x = \ln^{-1} x = \exp x$.

Note. The domain of e^x is $(-\infty, \infty)$ (the same as the range of $\ln x$) and the range of e^x is $(0, \infty)$ (the same as the domain of $\ln x$).

Note. Since we have defined e^x as the inverse of $\ln x$, we immediately have:

$$\begin{aligned} e^{\ln x} &= x \text{ for } x \in (0, \infty) \\ \ln(e^x) &= x \text{ for all } x. \end{aligned}$$

Definition. For any numbers $a > 0$ and for any real x,

$$a^x = e^{x \ln a}.$$

Note. The above definition for the first time allows us to deal with irrational exponents. In fact, this is the first time you have rigorously dealt with irrational exponents. Remember that for any positive real number x, $x^{p/q}$ for p and q integers is defined as $x^{p/q} = \sqrt[q]{x^p} = (\sqrt[q]{x})^p$. For an irrational exponent s, x^s is defined in terms of natural logarithms and exponentials: $x^s = e^{s \ln x}$. We might expect to define exponentiation with irrational exponents in terms of limits. For example, we might define $3^{\sqrt{2}}$ as the limit of the rational powers of 3 in which the exponents approach $\sqrt{2}$:

$$3^1, 3^{1.4}, 3^{1.41}, 3^{1.414}, 3^{1.4142}, 3^{1.41421}, 3^{1.414213}, 3^{1.4142135}, 3^{1.41421356}, \ldots$$

We will see that this idea is correct, when we deal with sequences (and the fact that exponential functions are continuous).

Theorem 3. For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$.
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1x_2} = (e^{x_2})^{x_1}$

Note. The proof of Theorem 3 is based on the definition of $y = e^x$ in terms of $x = \ln y$ and properties of the natural logarithm function.

Example. Page 493 numbers 8 and 12.

Theorem. We have

$$\frac{d}{dx}\left[e^x\right] = e^x.$$

Proof. As in Theorem 1, we know that with $y = e^x$ we have

$$\ln y = \ln e^x = x.$$

Then

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}[x]$$

$$\frac{1}{y}\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = y = e^x.$$

Q.E.D.

Note. By combining the previous theorem with the Chain Rule we have

$$\frac{d}{dx}\left[e^u\right] = e^u\left[\frac{du}{dx}\right].$$

Example. Page 493 numbers 34 and 36.

Theorem. We have

$$\int e^u\,du = e^u + C.$$

Proof. This is just a statement of the previous theorem in integral form. *Q.E.D.*

Examples. Page 493 number 54 and page 494 number 64.

Theorem 4. We can find e as a limit:

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

Proof. Let $f(x) = \ln x$. Then $f'(x) = 1/x$ and $f'(1) = 1$. Now by the definition of derivative:

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{(1+x) - f(1)}{x} \\ &= \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x) \\ &= \lim_{x \to 0} \ln(1+x)^{1/x} \\ &= \ln\left(\lim_{x \to 0} (1+x)^{1/x}\right) \text{ since } \ln x \text{ is continuous.} \end{aligned}$$

Therefore since $f'(1) = 1$ we have

$$\ln\left(\lim_{x \to 0} (1+x)^{1/x}\right) = 1.$$

Since $\ln e = 1$ and $\ln x$ is one-to-one,

$$\lim_{x \to 0} (1+x)^{1/x} = e.$$

Q.E.D.

Note. We can use the previous theorem to find that

$$e \approx 2.7\ 1828\ 1828\ 45\ 90\ 45\ 9.$$

Theorem. Power Rule (General Form). If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\frac{d}{dx}\left[u^n\right] = nu^{n-1}\frac{du}{dx}.$$

Proof. First,

$$\begin{aligned}
\frac{d}{dx}\left[x^n\right] &= \frac{d}{dx}\left[e^{n\ln x}\right] \\
&= e^{n\ln x}\frac{d}{dx}\left[n\ln x\right] \text{ by the Chain Rule} \\
&= x^n\frac{n}{x} \\
&= nx^{n-1}.
\end{aligned}$$

Combining this with the Chain Rule gives the result. *Q.E.D.*

Example. Page 493 Example 9.