Chapter 7. Transcendental Functions

7.4. a^x and $\log_a x$

Recall. For any numbers $a > 0$ and for any real x, $a^x = e^{x \ln a}$.

Theorem. If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}[a^u] = a^u \ln a \left[\frac{du}{dx}\right].$$

Proof. First

$$\begin{aligned} \frac{d}{dx}[a^x] &= \frac{d}{dx}\left[e^{x \ln a}\right] \\ &= e^{x \ln a}\left[\frac{d}{dx}[x \ln a]\right] \\ &= a^x \ln a. \end{aligned}$$

Combining this result with the Chain Rule yields the theorem. *Q.E.D.*

Note. Notice that the previous theorem implies that $\frac{d}{dx}[a^x] = a^x \ln a$. With $a = e$, we have the special case $\frac{d}{dx}[e^x] = e^x(1) = e^x$. This is what is *natural* about e When you first meet the natural exponential and logarithmic functions in algebra, it is hard to understand what is NATURAL about them. That is because the "natural-ness" is a calculus property (namely this differentiation property).

Example. Page 500 number 14.

Theorem. We have for differentiable u and $a > 0$

$$\int a^u \, du = \frac{a^u}{\ln a} + C.$$

Proof. This is the integral version of the previous theorem. *Q.E.D.*

Example. Page 501 number 52.

Definition. For any $a > 0$, $a \neq 1$, define $\log_a x$ as the inverse of a^x.

Note. It follows from the definitions that

$$a^{\log_a x} = x \quad \text{for all } x > 0$$

$$\log_a(a^x) = x \quad \text{for all } x.$$

Theorem. To convert logarithms from one base to another, we have the formula:

$$\log_a x = \frac{\ln x}{\ln a}.$$

Proof. From definition,

$$\begin{aligned} a^{\log_a(x)} &= x \\ \ln a^{\log_a(x)} &= \ln x \\ \log_a(x) \cdot \ln a &= \ln x \\ \log_a x &= \frac{\ln x}{\ln a}. \end{aligned}$$

Q.E.D.

Note. Of course, the usual laws of logarithms follow: For any $x > 0$ and $y > 0$

1. Product Rule: $\log_a xy = \log_a x + \log_a y$,

2. Quotient Rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$,

3. Reciprocal Rule: $\log_a \frac{1}{y} = -\log_a y$,

4. Power Rule: $\log_a x^y = y \log_a x$.

Example. Page 500 numbers 2, 6, and 10.

Theorem. Differentiating a logarithm base a gives:

$$\frac{d}{dx}[\log_a u] = \frac{1}{\ln a}\frac{1}{u}\frac{du}{dx}.$$

Proof. This follows easily:

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right] = \frac{1}{\ln a}\frac{d}{dx}[\ln x] = \frac{1}{\ln a}\frac{1}{x}.$$

Combining this result with the Chain Rule gives the theorem. *Q.E.D.*

Examples. Page 500 numbers 32 and 70.

Theorem. If $\lim_{x\to a} \ln f(x) = L$ then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$

Here, a may be finite or infinite.

Note. The proof of the previous theorem follows from the continuity of the exponential function at every real number. This result allows us to extend L'Hopital's Rule to indeterminate forms 1^∞, 0^0, and ∞^0.

Example. Page 502 numbers 99 and 100.