

## Chapter 7. Transcendental Functions

### 7.5 Exponential Growth and Decay

**Note.** Suppose we are interested in a quantity  $y$  (population, radioactive element, money) that increases or decreases at a rate proportional to the amount present. If we also know the amount present at time  $t = 0$ , say  $y_0$ , we can find  $y$  as a function of  $t$  by solving the following initial value problem.

$$\text{Differential equation: } \frac{dy}{dt} = ky$$

$$\text{Initial condition: } y = y_0 \text{ when } t = 0.$$

The constant function  $y = 0$  is a solution of the differential equation but we usually aren't interested in that solution. To find nonzero solutions, we separate the variables and integrate.

$$\frac{dy}{y} = k dt$$

$$\ln |y| \in kt + C$$

$$e^{\ln |y|} = e^{kt+c} \text{ where } c \in C$$

$$|y| = e^c e^{kt} \text{ where } c \in C$$

$$y = \pm e^c e^{kt}$$

$$y = Ae^{kt} \text{ where } A = \pm e^c.$$

By allowing  $A$  to take on the value 0 in addition to all the possible values we can include the solution  $y = 0$ . Therefore the general solution to the given differential equation is  $y = Ae^{kt}$  where  $A$  is an arbitrary constant.

**Note.** If  $y$  changes at a rate proportional to the amount present ( $dy/dx = ky$ ) and  $y = y_0$  when  $t = 0$ , then

$$y = y_0e^{kt},$$

where  $k > 0$  represents growth and  $k < 0$  represents decay. This is call the *Law of Exponential Change* and  $k$  is the *rate constant* of the equation.

**Example.** Page 509 number 8.

**Note.** Suppose that  $A_0$  dollars are invested at a fixed annual interest rate  $r$ . If interest is added to the account  $k$  times a year, the amount of money present after  $t$  years is

$$A(t) = A_0 \left(1 + \frac{r}{k}\right)^{kt}.$$

For example, if the interest is compounded monthly, then  $k = 12$ . If the interest is compounded more and more frequently, then we can calculate a limit as  $t$  approaches infinity to get *continuously compounded* interest.

One can show (see page 505)

$$\lim_{k \rightarrow \infty} \left(1 + \left(\frac{r}{k}\right)\right)^k = e^r.$$

Therefore we get

$$A_0 e^{rt} = \lim_{k \rightarrow \infty} A_0 \left(1 + \frac{r}{k}\right)^{kt}.$$

So in the case of continuously compounded interest at a rate of  $r$  (per year) for time  $t$  (years) and an initial amount  $A_0$ , the amount present is

$$A(t) = A_0 e^{rt}.$$

**Note.** When an atom emits some of its mass as radiation, the remainder of the atom reforms to make an atom of some new element. This process of radiation and change is *radioactive decay*, and an element whose atoms go spontaneously through this process is *radioactive*. For example, radioactive carbon-14 decays into nitrogen.

Experiments have shown that at any given time, the rate at which a radioactive element decays (as measured by the number of nuclei that change per unit of time) is approximately proportional to the number of radioactive nuclei present. Thus, the decay of a radioactive element is described by the equation  $dy/dt = -ky$ ,  $k > 0$ . If  $y_0$  is the number of radioactive nuclei present at time zero, the number still present at any

later time will be

$$y = y_0 e^{-kt}, k > 0.$$

**Definition.** The *half-life* of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay.

**Note.** We can calculate half-life by asking  $t = ?$  when  $y = y_0/2$ . This gives  $y_0/2 = y_0 e^{-kt}$  or  $1/2 = e^{-kt}$ . By taking logarithms of both sides of this last equation, we get  $\ln(1/2) = \ln e^{-kt} = -kt$ . From this it follows that  $t = -(\ln 1/2)/k = (\ln 2)/k$ .

**Example.** The following radioactive isotopes are commonly used for determining ages of rocks:

Isotope	Half-Life	Daughter Product
K-40	1.3 billion years	Ar-40
U-238	4.5 billion years	Pb-206
U-235	713 million years	Pb-207
Th-232	14.1 billion years	Pb-208
Rb-87	49 billion years	Sr-87

When a rock is formed, it contains U-238 and no Pb-206. After some time, the rock contains only 49% of the original amount of U-238 (the rest having decayed into Pb-206). How old is the rock?

**Note.** If  $H$  is the temperature of an object in an environment of temperature  $H_S$ , then according to *Newton's Law of Cooling*, the objects temperature changes at a rate proportional to the difference of  $H$  and  $H_S$ . That is,

$$\frac{dH}{dt} = -k(H - H_S).$$

If  $H_0$  is the initial temperature of the object, then we get

$$H - H_S = (H_0 - H_S)e^{-kt}.$$

**Example.** Page 511 number 24.