Chapter 7. Transcendental Functions

7.7. Inverse Trigonometric Functions

Recall. The six inverse trigonometric functions are defined as follows:

1. $y = \cos^{-1} x$ if and only if $\cos y = x$ and $y \in [0, \pi]$.

2. $y = \sin^{-1} x$ if and only if $\sin y = x$ and $y \in [-\pi/2, \pi/2]$.

3. $y = \tan^{-1} x$ if and only if $\tan y = x$ and $y \in (-\pi/2, \pi/2)$.

4. $y = \sec^{-1} x$ if and only if $\sec y = x$ and $y \in [0, \pi/2) \bigcup (\pi/2, \pi]$.

5. $y = \csc^{-1} x$ if and only if $\csc y = x$ and $y \in [-\pi/2, 0) \bigcup (0, \pi/2]$.

6. $y = \cot^{-1} x$ if and only if $\cot y = x$ and $y \in (0, \pi)$.

For all appropriate x values:

$$\begin{aligned}
\sec^{-1} x &= \cos^{-1}(1/x) \\
\csc^{-1} x &= \sin^{-1}(1/x) \\
\cot^{-1} x &= \pi/2 - \tan^{-1} x.
\end{aligned}$$

Note. The graphs of the six inverse trig functions are:

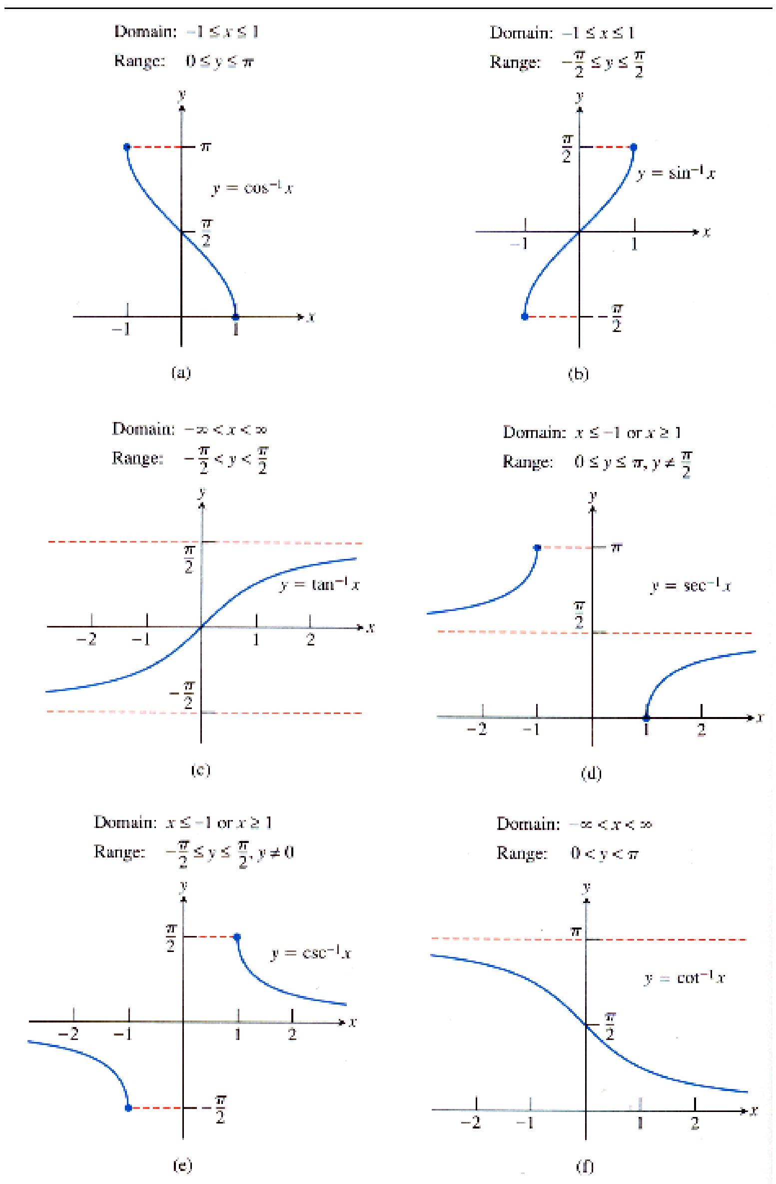

Figure 7.15 Page 519

Example. Page 530 number 4, Page 531 numbers 14, 28 and 38.

Theorem. We differentiate $\sin^{-1}$ as follows:

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

where $|u| < 1$.

Proof. We know that if $y = \sin^{-1} x$ then (for appropriate domain and range values) $\sin y = x$ and so by implicit differentiation

$$\begin{aligned} \frac{d}{dx}\left[\sin y\right] &= \frac{d}{dx}\left[x\right] \\ \cos y\left[\frac{dy}{dx}\right] &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y}. \end{aligned}$$

Since we have restricted y to the interval $[-\pi/2, \pi/2]$, we know that $\cos y \geq 0$ and so $\cos y = +\sqrt{1-(\sin y)^2} = \sqrt{1-x^2}$. Making this substitution we get

$$\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}.$$

The theorem then follows from the Chain Rule. *Q.E.D.*

Example. Page 531 number 58.

Theorem. We differentiate $\tan^{-1}$ as follows:

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{1+u^2}\left[\frac{du}{dx}\right].$$

Proof. We know that if $y = \tan^{-1} x$ then (for appropriate domain and range values) $\tan y = x$ and so by implicit differentiation

$$\begin{aligned}
\frac{d}{dx}[\tan y] &= \frac{d}{dx}[x] \\
\sec^2 y\left[\frac{dy}{dx}\right] &= 1 \\
\frac{dy}{dx} &= \frac{1}{\sec^2 y} \\
&= \frac{1}{1+(\tan y)^2} \\
&= \frac{1}{1+x^2}.
\end{aligned}$$

The theorem then follows from the Chain Rule. *Q.E.D.*

Example. Page 531 number 62.

Theorem. We differentiate $\sec^{-1}$ as follows:

$$\frac{d}{dx}\left[\sec^{-1} u\right] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

where $|u| > 1$.

Proof. We know that if $y = \sec^{-1} x$ then (for appropriate domain and range values) $\sec y = x$ and so by implicit differentiation

$$\begin{aligned} \frac{d}{dx}[\sec y] &= \frac{d}{dx}[x] \\ \sec y \tan y \left[\frac{dy}{dx}\right] &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec y \tan y}. \end{aligned}$$

We now need to express this last expression in terms of x. First, $\sec y = x$ and $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2-1}$. Therefore we have

$$\frac{d}{dx}\left[\sec^{-1}\right] = \pm\frac{1}{x\sqrt{x^2-1}}.$$

Notice from the graph of $y = \sec^{-1} x$ above, that the slope of this function is positive where ever it is defined. So

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \begin{cases} +\frac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \\ -\frac{1}{x\sqrt{x^2-1}} & \text{if } x < -1. \end{cases}$$

Notice that if $x > 1$ then $x = |x|$ and if $x < -1$ then $-x = |x|$. Therefore

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{|x|\sqrt{x^2-1}}.$$

The Theorem then follows from the Chain Rule. *Q.E.D.*

Note. We can use the following identities to differentiate the other three inverse trig functions:

$$\begin{aligned} \cos^{-1} x &= \pi 2 - \sin^{-1} x \\ \cot^{-1} x &= \pi/2 - \tan^{-1} x \\ \csc^{-1} x &= \pi/2 - \sec^{-1} x \end{aligned}$$

We then see that the only difference in the derivative of an inverse trig function and the derivative of the inverse of its cofunction is a negative sign. In summary, that is (Table 7.3 page 528):

1. $\dfrac{d}{dx}\left[\sin^{-1} u\right] = \dfrac{du/dx}{\sqrt{1-u^2}}, |u| < 1$
2. $\dfrac{d}{dx}\left[\cos^{-1} u\right] = -\dfrac{du/dx}{\sqrt{1-u^2}}, |u| < 1$
3. $\dfrac{d}{dx}\left[\tan^{-1} u\right] = \dfrac{du/dx}{1+u^2}$
4. $\dfrac{d}{dx}\left[\cot^{-1} u\right] = -\dfrac{du/dx}{1+u^2}$
5. $\dfrac{d}{dx}\left[\sec^{-1} u\right] = \dfrac{du/dx}{|u|\sqrt{u^2-1}}, |u| > 1$
6. $\dfrac{d}{dx}\left[\csc^{-1} u\right] = \dfrac{-du/dx}{|u|\sqrt{u^2-1}}, |u| < 1$

Example. Page 531 number 60.

Theorem. For constant $a \neq 0$ we have

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (where $u^2 < a^2$)
2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (where $|u| > a > 0$)

Proof. These are merely the integral versions of the results in Table 7.3 (with a substitution to compensate for the a value). *Q.E.D.*

Examples. Page 532 numbers 74, 96, page 534 number 150.