Chapter 8. Techniques of Integration

8.4 Trigonometric Integrals

Note. Since the derivative of $\sin x$ is $\cos x$ and the derivative of $\cos x$ is $-\sin x$, an integral of the form $\displaystyle\int \sin^m x \cos^n x\, dx$ can be evaluated if we can eliminate all but one $\sin x$ or all but one $\cos x$. This can frequently be accomplished with identities.

Case 1. If m is odd, we write $m = 2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x.$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\, dx$ equal to $-d[\cos x]$.

Case 2. If m is even and n is odd in $\int \sin^m x \cos^n x\, dx$, we write n as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1-\sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x\, dx$ equal to $d[\sin x]$.

Case 3. If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Example. Page 585 number 6, page 586 number 12, 8.

Note. The identities $\cos^2 x = (1 + \cos 2x)/2$ and $\sin^2 x = (1 - \cos 2x)/2$ can be used to eliminate square roots.

Example. Page 586 number 18.

Note. The identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$ (along with the corresponding cofunction identities) can be used to integrate powers of $\tan x$ or $\sec x$.

Example. Page 584 Example 6 and page 586 number 32.

Note. We can also evaluate integrals of the forms

$$\int \sin mx \sin nx \, dx \quad \int \sin mx \cos nx \, dx \quad \int \cos mx \cos nx \, dx$$

using the following trig identities, which follow from the addition formluas for sin and cos:

$$\begin{aligned}
\sin mx \sin nx &= \frac{1}{2}[\cos(m-n)x - \cos(m+n)x] \\
\sin mx \cos nx &= \frac{1}{2}[\sin(m-n)x + \sin(m+n)x] \\
\cos mx \cos nx &= \frac{1}{2}[\cos(m-n)x + \cos(m+n)x].
\end{aligned}$$

Example. Page 586 number 34.