

Chapter 8. Techniques of Integration

8.5 Trigonometric Substitution

Note. In this section, we consider integrals involving expressions of the forms

1. $a^2 + x^2$

2. $a^2 - x^2$

3. $x^2 - a^2$

in “inconvenient” places (such as under square root radicals). We make the following substitutions in each case:

x	expression E	E becomes	dx
$a \tan \theta$	$a^2 + x^2$	$a^2 \sec^2 \theta$	$a \sec^2 \theta d\theta$
$a \sin \theta$	$a^2 - x^2$	$a^2 \cos^2 \theta$	$a \cos \theta d\theta$
$a \sec \theta$	$x^2 - a^2$	$a^2 \tan^2 \theta$	$a \sec \theta \tan \theta d\theta$

Note. After we convert from x values (say) to trig functions of θ (say), we will eventually need to reverse the process and convert back to x 's. We will do so using a “reference triangle.” We will find:

$$x = a \tan \theta \text{ requires } \theta = \tan^{-1} \left(\frac{x}{a} \right) \text{ with } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right],$$

$$x = a \sin \theta \text{ requires } \theta = \sin^{-1} \left(\frac{x}{a} \right) \text{ with } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right],$$

$$x = a \sec \theta \text{ requires } \theta = \sec^{-1} \left(\frac{x}{a} \right) \text{ with } \begin{cases} \theta \in \left[0, \frac{\pi}{2} \right), & \frac{x}{a} \geq 1 \\ \theta \in \left(\frac{\pi}{2}, \pi \right], & \frac{x}{a} \leq -1 \end{cases}$$

Examples. Page 591 numbers 16, 6, 22.

Example. Find the arclength of the graph of $y = x^2$ for $x \in [0, 1]$.