Chapter 9. Further Applications of Integration

9.1 Slope Fields and Separable Differential Equations

Definition. A *first-order differential equation* is a relation

$$\frac{dy}{dx} = f(x, y)$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane. A *solution* of this equation is a differentiable function $y = y(x)$ defined on an interval of x-values such that

$$\frac{d}{dx}[y(x)] = f(x, y(x))$$

on that interval. The initial condition that $y(x_0) = y_0$ amounts to requiring the solution curve $y = y(x)$ to pass though the point (x_0, y_0).

Example. Page 648 number 6.

Note. We can graph little hash-marks to indicate the slope $dy/dx = f(x, y)$ at various points in the $xy - plane$ to give some idea of the "flow" of a solution. Such a collection of hash-marks is called a *slope field* for the differential equation.

Example. The slope field for the differential equation $\frac{dy}{dx} = y - x$ is given below, along with the particular solution which passes through the point $(0, 2/3)$.

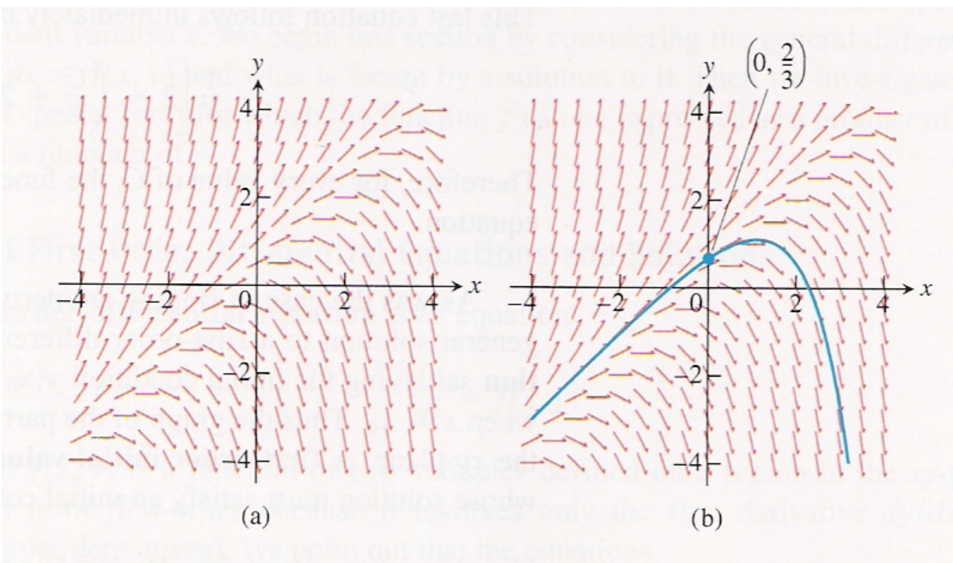

Figure 9.2, page 644.

Example. Page 649 number 24.

Definition. The equation $y' = f(x, y)$ is *separable* if f can be expressed as a product of a function of x and a functions of y. The differential equation then has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Note. If $h(y) \neq 0$, we can *separate the variables* by dividing both sides by h, obtaining

$$\begin{aligned} \frac{1}{h(y)}\frac{dy}{dx} &= g(x) \\ \int \frac{1}{h(y)}\frac{dy}{dx}\,dx &= \int g(x)\,dx \\ \int \frac{1}{h(y)}\,dy &= \int g(x)\,dx \end{aligned}$$

(Notice that the last two lines claim that two *sets* are equal.) With x and y now separated, we simply integrate each side to get the solutions. We seek by expressing y either explicitly or implicitly as a function of x, up to an arbitrary constant.

Example. Page 648 number 12.