# Chapter 9. Further Applications of Integration

## 9.2 Linear First-Order Differential Equations

**Definition.** A first-order differential equation that **can be** written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P$ and $Q$ are functions of $x$, is a *linear* first-order equation and the above equation is the *standard form* of the D.E.

**Theorem.** The solution of the linear equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is

$$y \in \frac{1}{v(x)} \int v(x)Q(x)\,dx,$$

where

$$v(x) \in e^{\int P(x)\,dx}.$$

In the "formula" for $v(x)$, we can simply set $v$ equal to the exponentiation of ANY antiderivative of $P$.

**Proof.** First, we multiply both sides of the equation by some function $v$ (called an *integrating factor*) which will transform the left-hand side of the D.E. into the derivative of the product $v(x)y$ (this is a constraint on $v$ that we will deal with shortly):

$$\begin{aligned}
\frac{dy}{dx} + P(x) &= Q(x) \\
v(x)\frac{dy}{dx} + P(x)v(x)y &= v(x)Q(x) \\
\frac{d}{dx}\left[v(x) \cdot y\right] &= v(x)Q(x) \\
v(x) \cdot y &\in \int v(x)Q(x)\,dx \\
y &\in \frac{1}{v(x)}\int v(x)Q(x)\,dx.
\end{aligned}$$

Now we must deal with the constraint on $v$:

$$\begin{aligned}
\frac{d}{dx}\left[v \cdot y\right] &= v\frac{dy}{dx} + Pvy \\
v\frac{dy}{dx} + y\frac{dy}{dx} &= v\frac{dy}{dx} + Pvy \\
y\frac{dv}{dx} &= Pvy.
\end{aligned}$$

This last equation will hold if

$$\begin{aligned}
\frac{dy}{dx} &= Pv \\
\frac{dv}{v} &= P\,dx \\
\int \frac{dv}{v} &\in \int P\,dx
\end{aligned}$$

$$\begin{aligned}
\ln v &\in \int P\,dx \\
&\quad \text{(notice } v > 0 \text{ by hypothesis)} \\
e^{\ln v} &\in e^{\int P\,dx} \\
v &\in e^{\int P\,dx}.
\end{aligned}$$

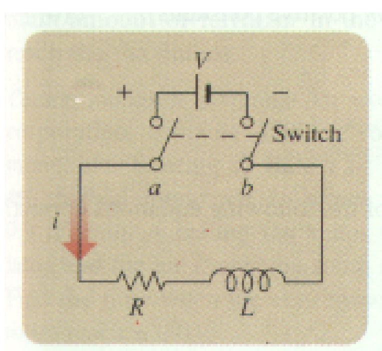
*Q.E.D.*

**Example.** Page 657 number 8.

**Note.** The diagram below represents an electrical circuit whose total resistance is a constant $R$ ohms and whose self-inductance, shown as a coil, is $L$ henries, also a constant. There is a switch whose terminal at $a$ and $b$ can be closed to connect a constant electrical source of $V$ volts. Ohm's Law, $V = RI$, has to be modified for such a circuit. The modified form is

$$L\frac{di}{dt} + Ri = V,$$

where $i$ is the intensity of the current in amperes and $t$ is the time in seconds. By solving this equation, we can predict how the current $i$ will flow after the switch is closed.



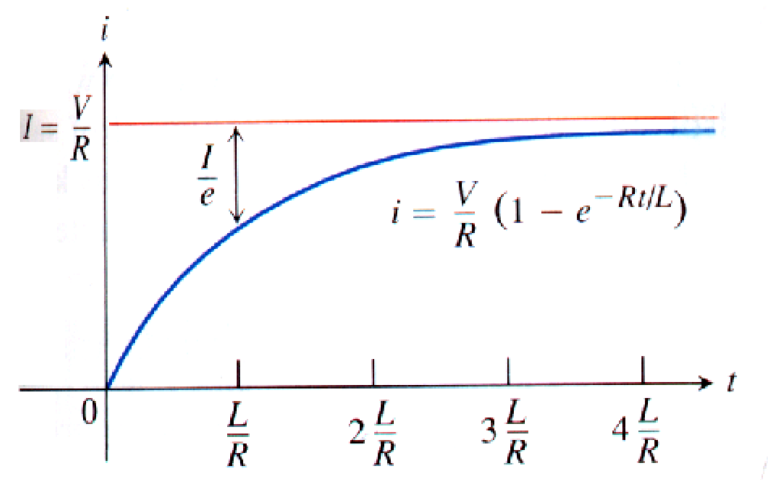

Page 654 Figure 9.5 (from 10th Edition)

**Example.** Page 659 number 32. Solution: $i = \dfrac{V}{R} - \dfrac{V}{R}e^{-(R/L)t}$.

**Note.** In the above problem, we have

$$\lim_{t\to\infty} i = \lim_{t\to\infty}\left(\frac{V}{R} - \frac{V}{R}e^{-(R/L)t}\right) = \frac{V}{R} - \frac{V}{R}\cdot 0 = \frac{V}{R}.$$

From the graph of the solution, we see why $i = V/R$ is called a *steady-state value*. In fact, the solution is expressed as the sum of a steady state solution $V/R$ and a *transient solution* $-(V/R)e^{-(R/L)t}$.




Page 655 Figure 9.6

**Note.** A chemical fluid (or gas) runs into a container containing a different fluid (or gas). The solution is then mixed so that the concentration of each fluid is the same throughout. At the same time, some of the fluid is removed from the container. The differential equation describing the process is based on the formula:

$$\begin{array}{c}\text{Rate of Change}\\ \text{of amount}\\ \text{in container}\end{array} = \left(\begin{array}{c}\text{rate at which}\\ \text{chemical arrives}\end{array}\right) - \left(\begin{array}{c}\text{rate at which}\\ \text{chemical departs.}\end{array}\right)$$

If $y(t)$ is the amount of chemical in the container at time $t$ and $V(t)$ is the total volume of liquid in the container at time $t$, then the departure of the chemical at time $t$ is

$$\begin{aligned}\text{Departure rate} &= \left(\begin{array}{c}\text{concentration in}\\ \text{container at time } t\end{array}\right)(\text{outflow rate})\\ &= \frac{y(t)}{V(t)}(\text{outflow rate}).\end{aligned}$$

Therefore the relevant equation is

$$\frac{dy}{dt} = (\text{chemical's arrival rate}) - \frac{y(t)}{V(t)}(\text{outflow rate}).$$

**Example.** Page 658 number 26.