

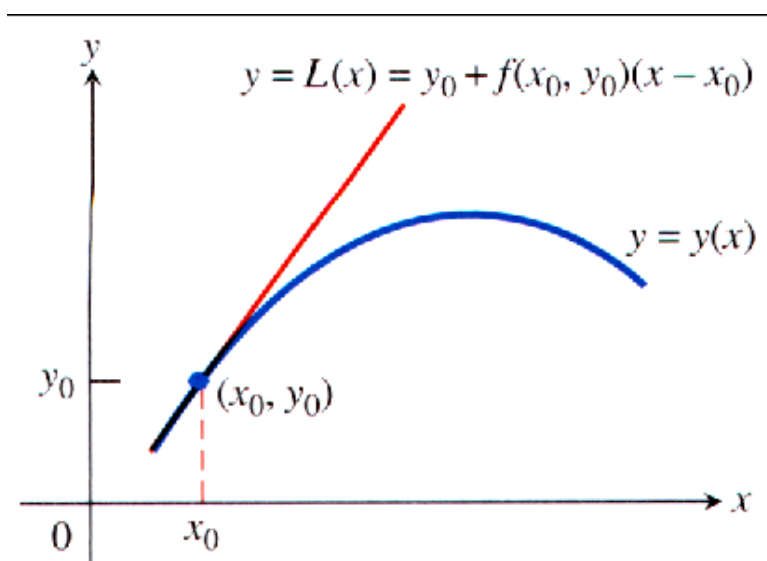
Chapter 9. Further Applications of Integration

9.3 Euler's Method

Note. (Euler's Method) Given a differential equation $dy/dx = f(x, y)$ and an initial condition $y(x_0) = y_0$, we can approximate the solution $y = y(x)$ by its linearization

$$L(x) = y(x_0) + y'(x_0)(x - x_0) \text{ or } L(x) = y_0 + f(x_0, y_0)(x - x_0).$$

The function $L(x)$ gives a good approximation to the solution $y(x)$ in a short interval about x_0 :

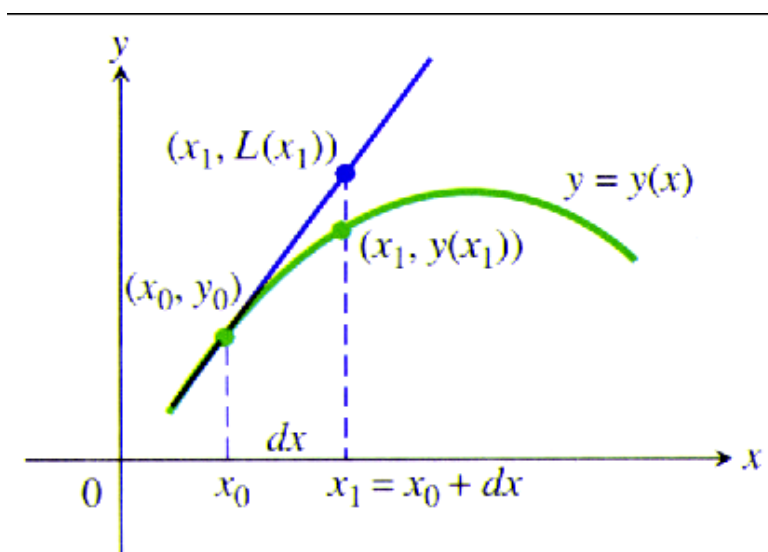


The basis of Euler's Method is to patch together a string of linearizations to approximate the curve over a long stretch.

We know the point (x_0, y_0) lies on the solution curve. Suppose that we specify a new value for the independent variable to be $x_1 = x_0 + dx$. If the increment dx is small, then

$$y_1 = L(x_1) = y_0 + f(x_0, y_0)dx$$

is a good approximation to the exact solution value $y = y(x_1)$. So from the point (x_0, y_0) , which lies exactly on the solution curve, we have obtained the point (x_1, y_1) which lies very close to the point $(x_1, y(x_1))$ on the solution curve:



Page 660 Figure 9.9

Using the point (x_1, y_1) and the slope $f(x_1, y_1)$ of the solution curve

through (x_1, y_1) we take a second step. Setting $x_2 = x_1 + dx$, we use the linearization of the solution curve through (x_1, y_1) :

$$y_2 = y_1 + f(x_1, y_1)dx.$$

This gives the next approximation (x_2, y_2) to values along the solution curve $y = y(x)$ with slope $f(x_2, y_2)$ to obtain the third approximation

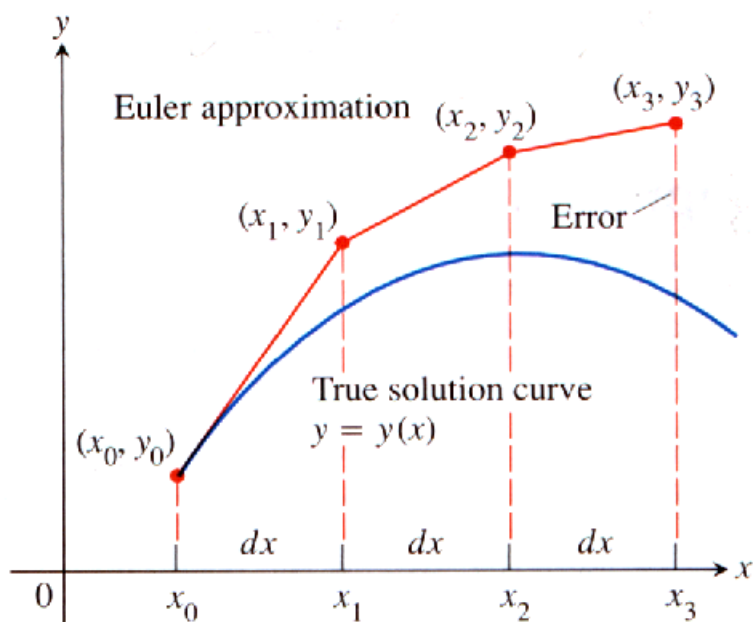
$$y_3 = y_2 + f(x_2, y_2)dx,$$

and so on. In general, we have

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})dx.$$

We are literally building an approximation to one of the solutions by following the direction of the “slope field” of the differential equation.

Note. We have $x_n = x_0 + n dx$.



Page 660 Figure 9.10

Example. Page 664 number 2.

Note. It might be tempting to reduce the step size more and more in Euler's Method in order to obtain better accuracy. However, this requires additional computation time and more importantly adds to the buildup of round-off errors due to the approximate representations of numbers. The analysis of errors and the investigation of methods to reduce it when making numerical calculations are important and are explored in a Numerical Analysis class. In fact, in such a class you will see that there are numerical methods more accurate and more sophisticated than Euler's method.

Note. We can improve on Euler's Method by taking an average of two slopes. We first estimate y_n as in the original Euler's Method, but denote it by z_n . We then take the average of $f(x_{n-1}, y_{n-1})$ and $f(x_n, z_n)$ in place of $f(x_{n-1}, y_{n-1})$ in the next step. Thus, we calculate the next approximation y_n using

$$\begin{aligned} z_n &= y_{n-1} + f(x_{n-1}, y_{n-1})dx \\ y_n &= y_{n-1} + \left(\frac{f(x_{n-1}, y_{n-1}) + f(x_n, z_n)}{2} \right) dx. \end{aligned}$$

Example. Page 663 Example 3. Use the improved Euler's Method to solve the I.V.P.

$$y' = 1 + y, \quad y(0) = 1,$$

on the interval $x \in [0, 1]$, starting at $x_0 = 0$ and taking $dx = 0.1$. Compare the approximations with the values of the exact solution $y = 2e^x - 1$. (See Table 9.3 on page 663.)