Calculus 2 Test 2 — Summer 2012

NAME KEY E-NUMBER ______

SHOW ALL WORK!!! Do not rely on the calculators. Include all necessary symbols (such as equal signs, limits, and constants of integration "+C"). The more details you show, the easier it will be to give you partial credit (if needed). Put your final answer in the box provided, or put a box around your final answer. Each numbered problem is worth 12 points.

p 459 #4

1. Evaluate $\int x^2 \sin x \, dx$.

let $u = x^2 \quad dv = \sin x \, dx$

$du = 2x\,dx \quad v = -\cos(x)$

$= (x^2)(-\cos(x)) - \int -\cos(x)\, 2x\,dx = -x^2\cos(x) + 2\int x\cos(x)\,dx$

let $u = x \quad dv = \cos(x)\,dx$

$du = dx \quad v = \sin(x)$

$= -x^2\cos(x) + 2\left(x\sin x - \int \sin x\,dx\right)$

$= -x^2\cos(x) + 2x\sin x - 2(-\cos(x)) + C$

$\boxed{-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C}$

p 466 #11

2. Evaluate $\int \sin^3 x \cos^3 x \, dx$.

$= \int \sin^2(x)\cos^3(x)\sin x\,dx = \int (1-\cos^2(x))\cos^3(x)\,\boxed{\sin(x)\,dx}$

let $u = \cos(x)$

$du = -\sin(x)\,dx$

$-du = \sin(x)\,dx$

$= \int (1-u^2)u^3(-du) = \int (u^5 - u^3)\,du$

$= \frac{1}{6}u^6 - \frac{1}{4}u^4 + C$

$= \frac{1}{6}\cos^6(x) - \frac{1}{4}\cos^4(x) + C$

$\boxed{\frac{1}{6}\cos^6(x) - \frac{1}{4}\cos^4(x) + C}$

p420 #25

3. Evaluate $\int \frac{dx}{(x^2-1)^{3/2}}$, $x > 1$.

$$\left[\begin{aligned} &\text{Let } x = \sec\theta \\ &dx = \sec\theta\tan\theta\, d\theta \end{aligned}\right.$$

$$= \int \frac{\sec\theta\tan\theta\, d\theta}{(\sec^2\theta - 1)^{3/2}} = \int \frac{\sec\theta\tan\theta}{\tan^3\theta}\, d\theta$$

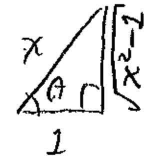
$$= \int \frac{\sec\theta}{\tan^2\theta}\, d\theta = \int \frac{1}{\cos\theta}\,\frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int \frac{\cos\theta\, d\theta}{\sin^2\theta}$$

$[$let $u = \sin\theta$, $du = \cos\theta\, d\theta$

$$= \int u^{-2}\,du = -u^{-1} + C = \frac{-1}{u} + C$$

$$= \frac{-1}{\sin\theta} + C. \text{ Consider}$$



so $\sec\theta = x$

$\cos\theta = \frac{1}{x}$

$\Rightarrow \sin\theta = \frac{\sqrt{x^2-1}}{x}$

$$= \frac{-x}{\sqrt{x^2-1}} + C$$

$$\boxed{\frac{-x}{\sqrt{x^2-1}} + C}$$

p479 #26

4. Evaluate $\int \frac{x^4+81}{x(x^2+9)^2}dx$. HINT: The partial fraction decomposition is $\frac{1}{x} - \frac{18x}{x^2+9}$.

$$\int \frac{x^4+81}{x(x^2+9)^2}\,dx = \int\left(\frac{1}{x} - \frac{18x}{x^2+9}\right)dx = \int \frac{1}{x}\,dx - 18\int \frac{x}{x^2+9}\,dx$$

let $u = x^2+9$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$= \int \frac{1}{x}\,dx - 18\int \frac{1}{u}\,\frac{du}{2}$$

$$= \ln|x| - 9\ln|u| + C$$

$$= \ln|x| - 9\ln|x^2+9| + C$$

$$\boxed{\ln|x| - 9\ln|x^2+9| + C}$$

p505 #13

5. Evaluate $\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$. Don't write anything wrong!

$$= \int_{-\infty}^{0} \frac{2x\,dx}{(x^2+1)^2} + \int_{0}^{\infty} \frac{2x}{(x^2+1)^2}\,dx = \lim_{a\to-\infty} \int_a^0 \frac{2x\,dx}{(x^2+1)^2} + \lim_{b\to\infty} \int_0^b \frac{2x\,dx}{(x^2+1)^2}$$

$$\left[\begin{array}{l} \text{let } u = x^2+1 \\ du = 2x\,dx \end{array}\right] = \lim_{a\to-\infty}\left(\int_{x=a}^{x=0} u^{-2}\,du\right) + \lim_{b\to\infty}\left(\int_{x=0}^{x=b} u^{-2}\,du\right)$$

$$= \lim_{a\to-\infty}\left(-u^{-1}\Big|_{x=a}^{x=0}\right) + \lim_{b\to\infty}\left(\left(-u^{-1}\right)\Big|_{x=0}^{x=b}\right)$$

$$= \lim_{a\to-\infty}\left(\frac{-1}{x^2+1}\Big|_a^0\right) + \lim_{b\to\infty}\left(\frac{-1}{x^2+1}\Big|_0^b\right) = \lim_{a\to-\infty}\left(-1 - \frac{-1}{a^2+1}\right)$$

$$+ \lim_{b\to\infty}\left(\left(\frac{-1}{b^2+1}\right) - (-1)\right) = (-1+(0)) + ((0)+1) = \boxed{0}$$

6. Solve $x\frac{dy}{dx} = x^2 + 3y$, $x > 0$. HINT: The solution of the linear equation $\frac{dy}{dx} + P(x)y = Q(x)$ is $y \in \frac{1}{v(x)}\int v(x)Q(x)\,dx$, where $v(x) \in e^{\int P(x)\,dx}$. In the "formula" for $v(x)$, we can simply set v equal to the exponentiation of ANY antiderivative of P.

p524 Eg 2

Well, $\frac{dy}{dx} = x + \frac{3}{x}y$ or $\frac{dy}{dx} - \frac{3}{x}y = x$. Let $P(x) = \frac{-3}{x}$, $Q(x) = x$.

Then $\int P(x)\,dx = \int \frac{-3}{x}\,dx = -3\ln|x| + C \Rightarrow v(x) = e^{-3\ln|x|} = |x|^{-3} = x^{-3}$ since $x > 0$.

So, $y \in \frac{1}{x^{-3}}\int (x^{-3})(x)\,dx = \frac{1}{x^{-3}}\int x^{-2}\,dx = \frac{1}{x^{-3}}(-x^{-1} + C)$.

Hence, $y = x^3\left(-\frac{1}{x} + k\right) = -x^2 + kx^3$ for some constant k.

$$\boxed{y = -x^2 + kx^3}$$

p533 #1a

7. A 66-kg cyclist on a 7-kg bicycle starts coasting on level ground at 9 m/sec. The value of k is 3.9 kg/sec. Use the equation for velocity of $v = v_0 e^{-(k/m)t}$ and an improper integral to determine the total distance the cyclists coasts. Don't worry about simplifying your answer.

Well, $m = 66+7 = 73$ kg, $v_0 = 9$ m/sec, and $k = 3.9$ kg/sec.

So $v = 9e^{-(3.9/73)t}$. The total distance travelled is

$$D = \int_0^\infty v(t)\,dt = \lim_{b\to\infty}\left(\int_0^b 9e^{-(3.9/73)t}\,dt\right) = \lim_{b\to\infty}\left(\frac{-73\times 9}{3.9}e^{-(3.9/73)t}\Big|_0^b\right)$$

$$= \lim_{b\to\infty}\left(\frac{-73\times 9}{3.9}e^{-(3.9/73)b} + \frac{73\times 9}{3.9}(1)\right) = \frac{73\times 9}{3.9}\text{ m}$$

$\boxed{\frac{73\times 9}{3.9}\text{ m} \approx 168.46\text{m}}$

p540 #2

8. Consider the autonomous differential equation $\frac{dy}{dx} = (y^2 - 4)$. Find the equilibria, construct a phase line, and sketch several solutions. Which equilibria are stable and which are unstable?

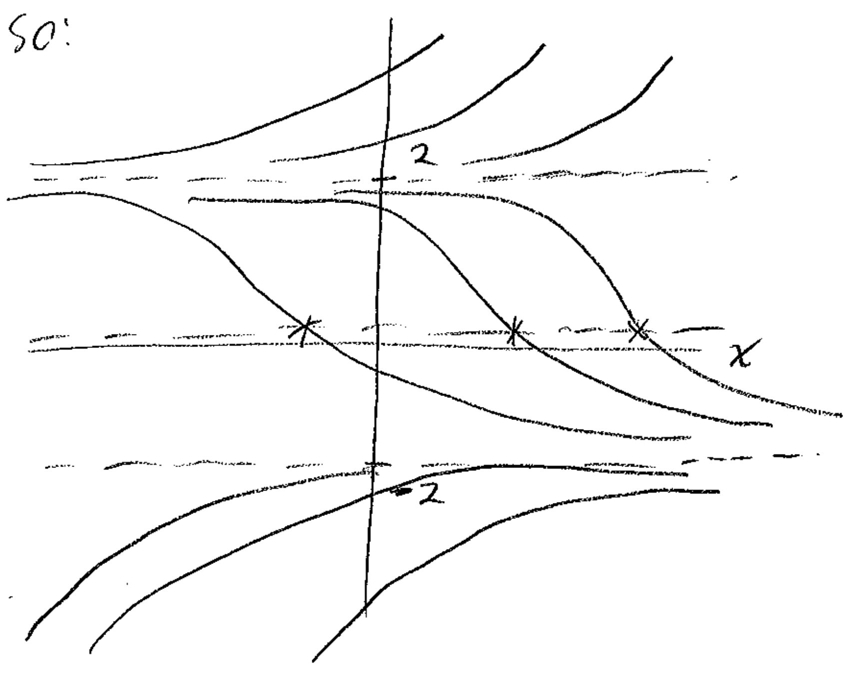
For equilibria, set $\frac{dy}{dx} = (y+2)(y-2) \stackrel{\text{set}}{=} 0 \Rightarrow$ $y=-2$ & $y=2$ are equilibria.

Next, $y'' = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}[y^2-4] = 2y[y'] = 2y(y+2)(y-2)$.

Consider

$-$	$+$	$-$	$+$	y''
$+$	$-$	$-$	$+$	y'
-2	0	2		y

SO:



SO, the equilibrium at $y=2$ is UNSTABLE and the equilibrium at $y=-2$ is STABLE p505H52

Bonus. Consider $\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$. Use the Limit Comparison Test to decide whether this integral converges or diverges.

Let $f(x) = \frac{1}{\sqrt{x^2-1}}$ and $g(x) = \frac{1}{\sqrt{x^2}} = \frac{1}{|x|} = \frac{1}{x}$ for $x \geq 2$.

Then, $L = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x^2-1}}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}}$

$= \lim_{x\to\infty} \frac{\sqrt{x^2}}{\sqrt{x^2-1}} = \sqrt{\lim_{x\to\infty}\left(\frac{x^2}{x^2-1}\right)} \overset{\infty/\infty}{=} \sqrt{\lim_{x\to\infty}\left(\frac{2x}{2x}\right)} = \sqrt{1} = 1.$

Next,

DIVERGES.

$\int_2^\infty g(x)\,dx = \int_2^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\left(\int_2^b \frac{1}{x}\,dx\right)$

$= \lim_{b\to\infty}\left(\ln|x|\Big|_2^b\right) = \lim_{b\to\infty}\left(\ln(b)-\ln(2)\right) = \infty.$

So $\int_2^\infty g(x)\,dx$ diverges and by the Limit Comparison Test,

$\int_2^\infty f(x)\,dx$ diverges.