Calculus 2 Test 3 — Summer 2012

NAME KEY E-NUMBER

SHOW ALL WORK!!! Do not rely on the calculators. Include all necessary symbols (such as equal signs and summation signs). Justify every claim you make (such as why a series converges or diverges). The more details you show, the easier it will be to give you partial credit (if needed). Put your final answer in the box provided, or put a box around your final answer. Each numbered problem is worth 12 points.

p 569 #24

1. Use series to express the number $1.\overline{414} = 1.414\,414\,414\,414\ldots$ as the ratio of two integers.

Well, $1.\overline{414} = 1 + 414 \sum_{n=1}^{\infty} \left(\frac{1}{1000^n}\right) = 1 + 414\left(\frac{1/1000}{1 - 1/1000}\right)$

$= 1 + 414\left(\frac{1/1000}{999/1000}\right) = 1 + \frac{414}{999} = \frac{999+414}{999} = \frac{1413}{999}$

$$\boxed{\frac{1413}{999}}$$

p 575 #4

2. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n+4}$. **Use the Integral Test** to determine if the series converges or diverges.

Define $f(x) = \frac{1}{x+4}$. Consider $\int_1^\infty f(x)\,dx = \int_1^\infty \left(\frac{1}{x+4}\right)dx$

$= \lim_{b\to\infty}\left(\int_1^b \left(\frac{1}{x+4}\right)dx\right) = \lim_{b\to\infty}\left(\ln|x+4|\Big|_1^b\right) = \lim_{b\to\infty}\left(\ln(b+4) - \ln(1+4)\right)$

$= \infty$. So by the Integral Test, the given series $\boxed{\text{DIVERGES.}}$

p580 #22

3. Consider the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$. **Use a Comparison Test** to determine if the series converges or diverges.

Let $a_n = \frac{n+1}{n^2\sqrt{n}}$ and $b_n = \frac{1}{n^{3/2}}$. Consider

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{n+1}{n^2\sqrt{n}}\right)\left(\frac{n^{3/2}}{1}\right) = \lim_{n\to\infty}\left(\frac{n^{5/2}+n^{3/2}}{n^{5/2}}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1.$$

So by the Limit Comparison Test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\frac{n+1}{n^2\sqrt{n}}\right)$ converges since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (it's a p-series with $p = \frac{3}{2} > 1$)

converges

p585 #37

4. Consider the series $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$. **Use the Ratio Test or Root Test** to determine if the series converges or diverges.

Well,

$$\rho = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{(n+1)!}{(2(n+1)+1)!}\right)\left(\frac{(2n+1)!}{n!}\right) = \lim_{n\to\infty} \frac{(n+1)!}{n!}\,\frac{(2n+1)!}{(2n+3)!}$$

$$= \lim_{n\to\infty} \frac{n+1}{(2n+3)(2n+2)} = 0.$$ So, by the Ratio Test, this series converges.

converges

5. Consider the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n+3}$. Does it converge absolutely, converge conditionally, or diverge?

p 591 #21

Well, $\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left(\frac{1}{n+3}\right) = 0$, so the series converges by the Alternating Series Test. Next, consider $\sum_{n=1}^{\infty}\left|(-1)^n \frac{1}{n+3}\right|$ $= \sum_{n=1}^{\infty}\left(\frac{1}{n+3}\right)$. Let $a_n = \frac{1}{n+3}$ and $b_n = \frac{1}{n}$. Then

$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{1}{n+3}\right)\left(\frac{n}{1}\right) = \lim_{n\to\infty}\left(\frac{n}{n+3}\right) = 1$, so $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\frac{1}{n+3}\right)$ diverges since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges (it's the harmonic series).

So the original series does not converge absolutely.

$\boxed{\text{conditional convergence}}$

6. Consider the power series $\sum_{n=0}^{\infty}\frac{3^n x^n}{n!}$. Find the radius of convergence.

p600 #12

By the Ratio Test for absolute convergence:

$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}x^{n+1}}{(n+1)!}\right|\left|\frac{n!}{3^n x^n}\right| = \lim_{n\to\infty}\left(\frac{3|x|}{n+1}\right) = 0.$

So the series converges for all x and $R = \infty$.

$\boxed{R = \infty}$

7. Find the Maclaurin series (that is, the Taylor series at $x = 0$) for $f(x) = \sin x$. Show details and the computation of the n^{th} derivative.

like p 606 # 2

Well: $f(x) = \sin(x)$ & $\sin(0) = 0$

$f'(x) = \cos(x)$ & $\cos(0) = 1$

$f''(x) = -\sin(x)$ & $-\sin(0) = 0$

$f'''(x) = -\cos(x)$ & $-\cos(0) = -1$

$f^{(4)}(x) = \sin(x)$ & $\sin(0) = 0$

In general, $f^{(4k)}(0) = 0$; $f^{(4k+1)}(0) = 1$, $f^{(4k+2)}(0) = 0$, $f^{(4k+3)}(0) = -1$

for k a whole number, so

$$f(x) = \sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + \frac{x}{1!} + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\boxed{\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}$$

8. Find the series representation of $\dfrac{x}{\sqrt[3]{1+x}}$. HINT: If we define $f(x) = (1+x)^m$, then we find that the Taylor series for f is $(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$, where we define (for **any** m) $\binom{m}{k} = \dfrac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$. This series converges for $|x| < 1$.

p 621 #10

Well, $$\frac{x}{\sqrt[3]{1+x}} = x \cdot (1+x)^{-1/3} = x \cdot \left(1 + \sum_{k=1}^{\infty} \binom{-1/3}{k} x^k\right) \text{ for } |x| < 1$$

$$= x + \sum_{k=1}^{\infty} \binom{-1/3}{k} x^{k+1} \text{ for } |x| < 1.$$

$$\boxed{x + \sum_{k=1}^{\infty} \binom{-1/3}{k} x^{k+1}}$$

p592
#53

Bonus. Determine how many terms should be used to estimate the sum of the entire series with an error of less than 0.001 for: $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2+3}$. You may express you answer in terms of square roots, if you like.

By the Alternating Series Remainder Theorem, we want

$u_n < 0.001$ or $\frac{1}{n^2+3} < \frac{1}{1000}$ or $n^2+3 > 1000$

or $n^2 > 997$ or $n > \sqrt{997} \approx 31.56$. So if we add up 31 terms (that's $\lfloor\sqrt{997}\rfloor$), then we are insured the error term will be less than u_{32} $(32 = \lceil\sqrt{997}\rceil)$ and $u_{32} < \frac{1}{1000}$.

$\boxed{31 \text{ (or } \lfloor\sqrt{997}\rfloor)}$