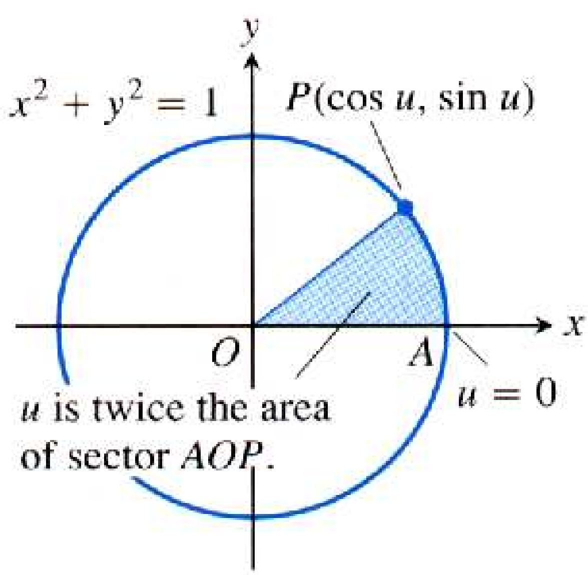
Chapter 6. Transcendental Functions and Differential Equations

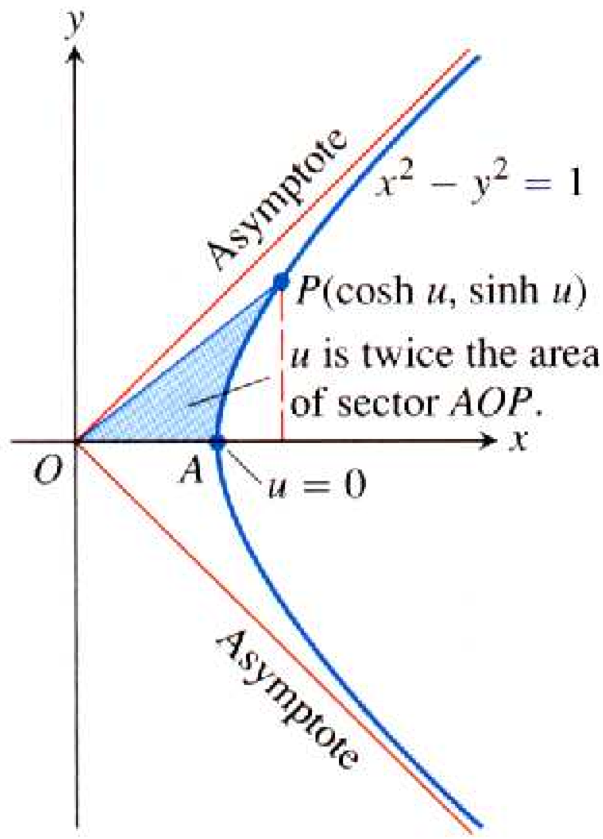
6.7 Hyperbolic Functions

Note. Recall that $\cos x$ and $\sin x$ are sometimes called the *circular functions*. This is because we can plot a point P on the circle $x^2 + y^2 = 1$ by letting $P = (\cos u, \sin u)$ where u is twice the area of the sector determined by $A = (1, 0)$, $O = (0, 0)$ and P:



Page 529 Figure 6.30b

Note. We can similarly define the *hyperbolic functions*. Consider the hyperbola $x^2 - y^2 = 1$. Choose a point P on the hyperbola and define u as twice the (signed) area determined by the sector $A = (1, 0)$, $O = (0, 0)$, and P. Now use the coordinates of P to define the hyperbolic trigonometric functions: $P = (\cosh u, \sinh u)$.



Page 529 Figure 6.30a

Note. We will use the exponential function to define the hyperbolic trig functions.

Definition. We define

Hyperbolic cosine of x: $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Hyperbolic sine of x: $\sinh x = \dfrac{e^x - e^{-x}}{2}$

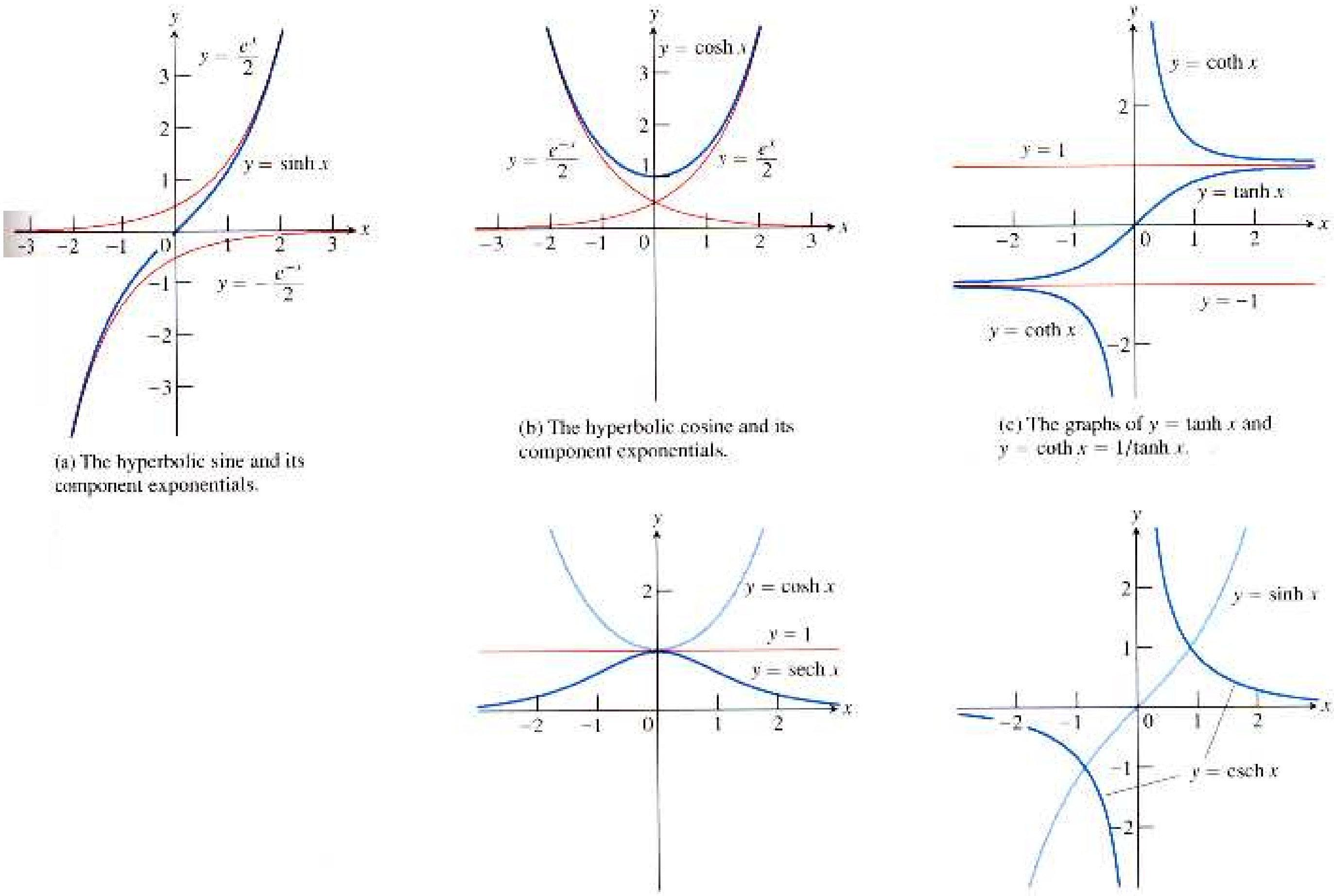
Hyperbolic tangent x: $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent of x: $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Hyperbolic secant of x: $\text{sech } x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

Hyperbolic cosecant of x: $\text{csch } x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

The graphs are:



(a) The hyperbolic sine and its component exponentials.

(b) The hyperbolic cosine and its component exponentials.

(c) The graphs of $y = \tanh x$ and $y = \coth x = 1/\tanh x$.

(d) The graphs of $y = \cosh x$ and $y = \operatorname{sech} x = 1/\cosh x$.

(e) The graphs of $y = \sinh x$ and $y = \operatorname{csch} x = 1/\sinh x$.

Page 521 Figure 6.26

Note. We have the following identities:

$$\cosh^2 x - \sinh^2 x = 1$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

Example. Page 526 number 2.

Theorem. (Table 6.13) We have the following differentiation properties:

$$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}[\cosh u] = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}[\tanh u] = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}[\coth u] = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}[\operatorname{sech} u] = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}[\operatorname{csch} u] = -\operatorname{csch} u \coth u \frac{du}{dx}$$

Example. Prove some of the results in the above theorem.

Theorem. (Table 6.14) We have the following integral properties:

$$\int \sinh u \, du = \cosh u + C$$

$$\int \cosh u \, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

Proof. These are just the integral versions of the results in Table 6.13.

Q.E.D.

Examples. Page 526 numbers 16 and 22.

Note. Since $\frac{d}{dx}[\sinh x] = \cosh x > 0$, then $\sinh x$ is an INCreasing function and so is one-to-one. The function $\cosh x$ is not one-to-one as we can see from the graph. The function $\operatorname{sech} x = 1/\cosh x$ is also not one-to-one. Therefore, to define the inverse functions of $\cosh x$ and $\operatorname{sech} x$, we must restrict the domains.

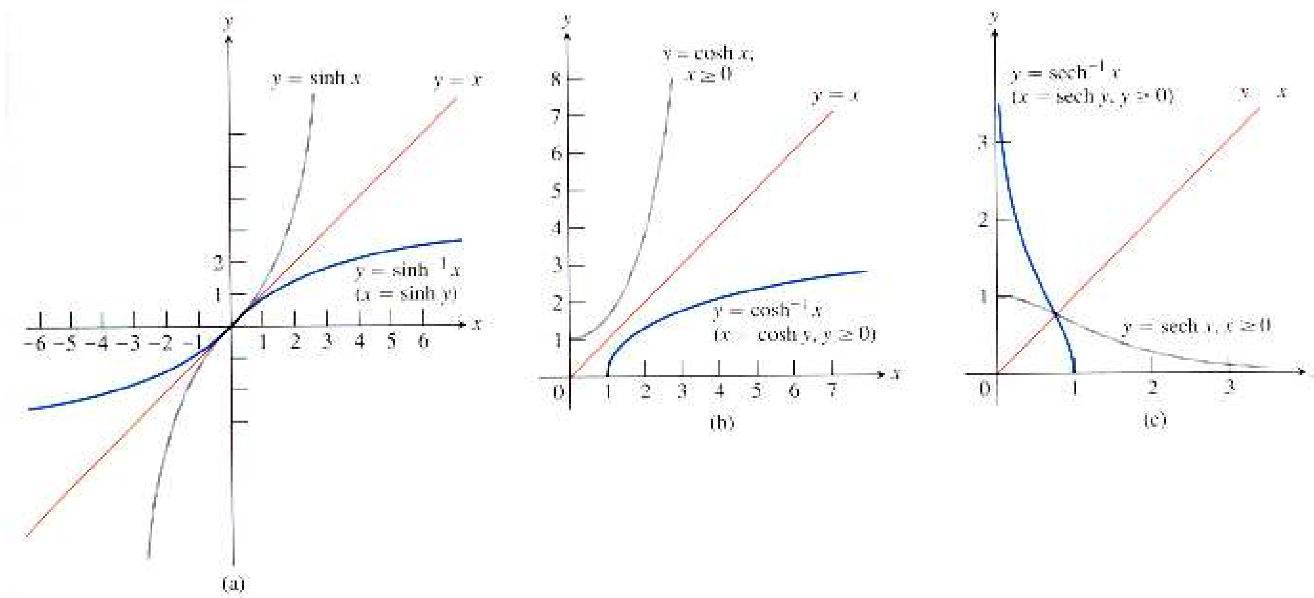
Definition. We define some inverse hyperbolic trig functions.

Define $y = \sinh^{-1} x$ if $x = \sinh y$. (The domain is then $x \in (-\infty, \infty)$.)

Define $y = \cosh^{-1} x$ if $x = \cosh y$ and $y \in [0, \infty)$. (The domain is then $x \in [1, \infty)$.)

Define $y = \operatorname{sech}^{-1} x$ if $x = \operatorname{sech} y$ and $y \in [0, \infty)$. (The domain is then $x \in (0, 1]$.)

Note. The graphs of the above defined three inverse hyperbolic trig functions are:

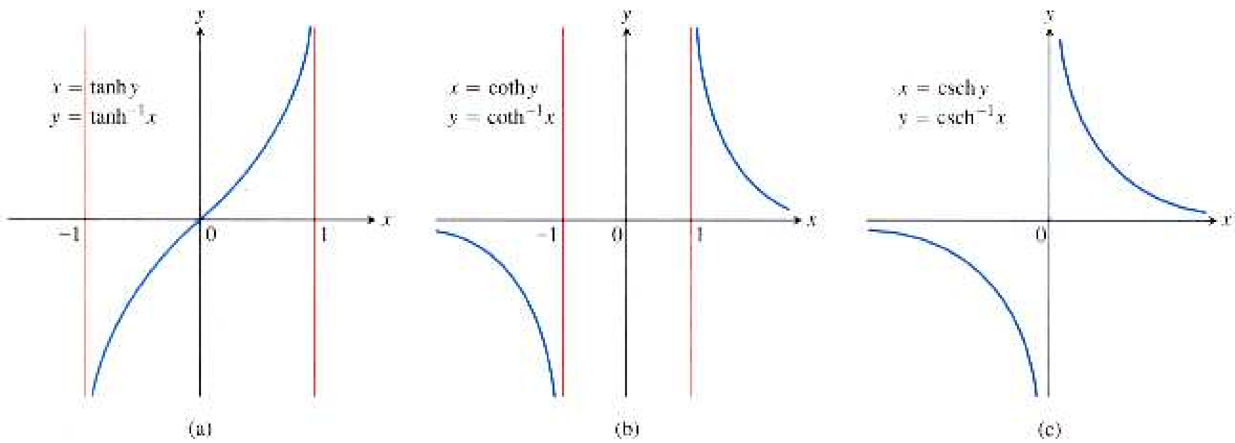


Page 523 Figure 6.27

Note/Definition. The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

$$y = \tanh^{-1} x,\ y = \coth^{-1} x,\ y = \text{ csch }^{-1} x.$$

The graphs of these functions are:



Page 524 Figure 6.28

Theorem. We can express the inverse hyperbolic trig functions in terms of the natural logarithm function as follows:

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}),\ \ x \in (-\infty, \infty).$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1}),\ \ x \in [1, \infty).$$

$$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x},\ \ x \in (-1, 1).$$

$$\text{sech }^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right),\ \ x \in (0, 1].$$

$$\text{csch }^{-1}x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right),\ \ x \in (-\infty, 0) \cup (0, \infty).$$

$$\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1},\ \ x \in (-\infty,-1)\cup(1,\infty).$$

Note. (Table 6.15) We can verify the following identities:

$$\text{sech}^{-1}x = \cosh^{-1}\frac{1}{x}$$

$$\text{csch}^{-1}x = \sinh^{-1}\frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1}\frac{1}{x}$$

Theorem. (Table 6.16) The inverse hyperbolic trig functions are differentiated as follows:

$$\frac{d}{dx}\left[\sinh^{-1}\right] = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cosh^{-1}\right] = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx},\ \ u \in (1,\infty)$$

$$\frac{d}{dx}\left[\tanh^{-1}\right] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx},\ \ u \in (-1,1)$$

$$\frac{d}{dx}\left[\coth^{-1}\right] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx},\ \ u \in (-\infty,-1)\cup(1,\infty)$$

$$\frac{d}{dx}\left[\text{sech}^{-1}\right] = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx},\ \ u \in (0,1)$$

$$\frac{d}{dx}\left[\text{csch}^{-1}\right] = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx},\ \ u \in (-\infty,0)\cup(0,\infty)$$

Example. Page 527 number 34.

Theorem. (Table 6.17) We have the following integrals involving inverse hyperbolic trig functions:

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C,\ a > 0$$

$$\int \frac{du}{\sqrt{a^2+u^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C,\ u > a > 0$$

$$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C & \text{if } u^2 > a^2 \end{cases}$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\,\text{sech}^{-1}\left(\frac{u}{a}\right) + C,\ 0 < u < a$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{a}{a}\,\text{csch}^{-1}\left|\frac{u}{a}\right| + C,\ u \neq 0$$

Examples. Page 527 number 74, page 528 number 79.

Corrected 1/18/2020