Chapter 8. Infinite Series

8.4 Series of Nonnegative Terms

Note. Given a series $\sum_{n=1}^{\infty} a_n$ we have two questions:

1. Does the series converge?

2. If it converges, what is its sum?

Corollary of Theorem 5. A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if its partial sums are bounded from above.

Proof. Theorem 5 (of section 8.2) implies that a monotonic increasing sequence which is bounded above must converge. A positive term series will have partial sums which form a monotonic increasing sequence. Since we have hypothesized that the sequence of partial sums is bounded, the result follows. *Q.E.D.*

Theorem. The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^\infty f(x)\,dx$ both converge or both diverge.

Proof. Since a finite number of terms does not affect the convergence of a series, we may assume that $N = 1$ without loss of generality. Under the hypotheses of f as continuous and decreasing, we can consider the following rectangles:

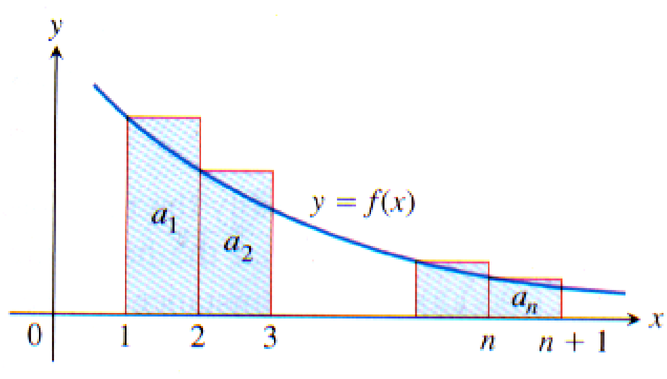

Figure 8.13a page 641

The areas of the rectangles are $a_1, a_2, a_3, \ldots, a_n$, and since f is decreasing, these rectangles are *circumscribed* over f and we have

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + \cdots + a_n.$$

If we consider *inscribed* rectangles, then we have:

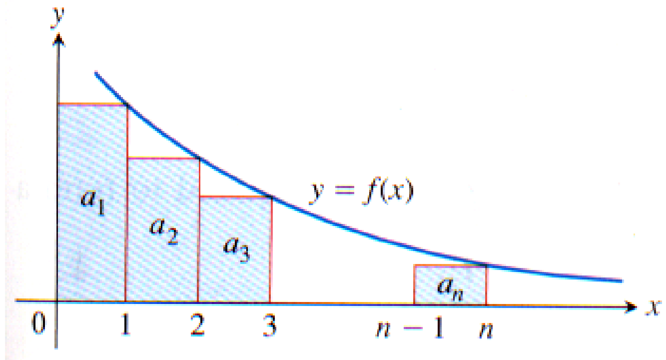

Figure 8.13b page 641

Excluding a_1, we see that

$$a_2 + a_3 + a_4 + \cdots + a_n \leq \int_1^n f(x)\,dx,$$

or that

$$a_1 + a_2 + a_3 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx.$$

Therefore we know that

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + a_3 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,x.$$

If $\displaystyle\int_1^{\infty} f(x)\,dx$ is finite, then the right-hand inequality shows that $\displaystyle\sum_{n=1}^{\infty} a_n$ is finite. If $\displaystyle\int_1^{\infty} f(x)\,dx$ is infinite, then the left-hand inequality shows that $\displaystyle\sum_{n=1}^{\infty} a_n$ is infinite. *Q.E.D.*

Example. Number 4 page 649.

Theorem. p**-Series**

A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

is called a *p-series*. A p series converges if $p > 1$ and diverges if $p \leq 1$.

Proof. We prove this using the Integral Test. First, suppose $p \neq 1$. Then

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{b\to\infty} \left(\int_1^b \frac{dx}{x^p} \right) = \lim_{b\to\infty} \left(\frac{x^{-p+1}}{-p+1} \right) \bigg|_1^b$$

$$= \lim_{b\to\infty} \left(\frac{1}{1-p}(b^{-p+1} - 1) \right) = \lim_{b\to\infty} \frac{1}{1-p} \left(\frac{1}{b^{p-1}} - 1 \right)$$

$$= \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore both the integral and the series converge if $p > 1$, and both diverge if $p < 1$. Next, suppose that $p = 1$. Then

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \left(\ln x \,\big|_1^b \right) = \lim_{b\to\infty} (\ln b - \ln 1) = \infty.$$

By the integral test, the series diverges when $p = 1$. *Q.E.D.*

Definition. The p-series with $p = 1$ is the *harmonic series*.

Note. Let's briefly explore the *rate* at which the harmonic series diverges. Example 3 on page 642 asks how many terms must we add in the harmonic series to get a partial sum greater than 20. Consider these two graphs:

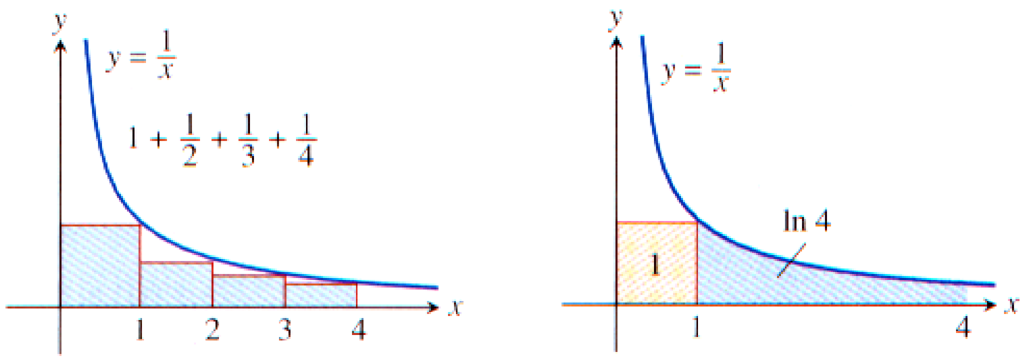

Figure 8.14 page 642.

We see that the 4th partial sum is less than $1 + \ln 4$, and in general the n^{th} partial sum will be less than $1 + \ln n$. Therefore we need *at least* $1 + \ln n > 20$, or $n > e^{19} \approx 178,482,301$. We can use a similar argument with circumscribed rectangles to see that the n^{th} partial sum is greater than $\ln(n+1)$, and so we find that to get the partial sum greater than 20, we would need *at most* $n = e^{20} - 1 \approx 485,165,194$.

Theorem. Direct Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with no negative terms.

(a) $\sum_{n=1}^{\infty} a_n$ converges if there is a convergent series $\sum_{n=1}^{\infty} c_n$ with $a_n \le c_n$ for all $n > N$, for some integer N.

(b) $\sum_{n=1}^{\infty} a_n$ diverges if there is a divergent series of nonnegative terms $\sum_{n=1}^{\infty} d_n$ with $a_n \ge d_n$ for all $n > N$, for some integer N.

Proof. For part **(a)**, the partial sums of $\sum_{n=1}^{\infty} a_n$ are bounded above by

$$M = a_1 + a_2 + \cdots + a_n + \sum_{n=N+1}^{\infty} c_n.$$

Therefore by the corollary to Theorem 5, the result holds.

For part **(b)**, the partial sums of $\sum_{n=1}^{\infty} a_n$ are not bounded above (for if they were, then the partial sums of $\sum_{n=1}^{\infty} d_n$ would be bounded and it would be convergent). Therefore $\sum_{n=1}^{\infty} a_n$ diverges. *Q.E.D.*

Example. Number 14 page 649.

Theorem. Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N a positive integer).

1. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c$, $0 < c < \infty$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

2. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

3. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Proof of (1). Since $c/2 > 0$, there exists an integer N such that for all $n > N$ we have $\left|\dfrac{a_n}{b_n} - c\right| < \dfrac{c}{2} \equiv \epsilon$. So for $n > N$ it follows that

$$-\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2},$$

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2},$$

$$\left(\frac{c}{2}\right) b_n < a_n < \left(\frac{3c}{2}\right) b_n.$$

If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges then $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3c}{2}\right) b_n$ converges and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges by the Direct Comparison Test. If $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} \left(\frac{c}{2}\right) b_n$ diverges and $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges by the Direct Comparison Test. *Q.E.D.*

Example. Number 16 page 649.

Theorem. The Ratio Test

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then

1. the series converges if $\rho < 1$,

2. the series diverges if $\rho > 1$ or if ρ is infinite, and

3. the test is inconclusive if $\rho = 1$ (that is, the series could diverge or converge — the Ratio Test tells us nothing).

Proof. **(a)** Let r be a number between ρ and 1. Then the number $\epsilon = r - \rho$ is positive. Since $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$, then there exists positive integer N such that for all $n \geq N$ we have a_{n+1}/a_n within ϵ of ρ. In particular, for $n \geq N$ we have $\dfrac{a_{n+1}}{a_n} < \rho + \epsilon = r$. That is,

$$\begin{aligned} a_{N+1} &< ra_N \\ a_{N+2} &< ra_{N+1} < r^2 a_N \\ a_{N+3} &< ra_{N+2} < r^3 a_N \end{aligned}$$

$$\vdots$$

$$a_{N+m} \; < \; r a_{N+m-1} < r^m a_N.$$

If we define the series

$$\sum_{n=1}^{\infty} c_n \equiv a_1 + a_2 + \cdots + a_{N-1} + a_N(1 + r + r^2 + \cdots),$$

then we see that this new series converges, for it is (eventually) a geometric series with ratio r between 0 and 1. Therefore by the Direct Comparison Test, the series $\sum_{n=1}^{\infty} a_n$ converges.

(b) With $1 < \rho \leq \infty$, we must eventually have (that is, for all $n \geq M$ where M is some positive integer) $\frac{a_{n+1}}{a_n} > 1$. That is, $0 < a_M < a_{M+1} < a_{M+2} < \cdots$. Therefore the sequence $\{a_n\}$ either diverges or has a limit greater than 0. So by the Test for Divergence, the series $\sum_{n=1}^{\infty} a_n$ diverges.

(c) Consider the two series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. For both series, $\rho = 1$, but the first series diverges, while the second series converges. *Q.E.D.*

Note. The Ratio Test (if applicable) is easier to use than the Direct Comparison Test. This is because you don't need to find a second series which has the appropriate behavior (in terms of convergence or divergence) and satisfies the appropriate inequalities.

Example. Number 24 page 649.

Theorem. The n^{th}-Root Test
Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \geq 0$ for $n \geq N$ and suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$. Then

(a) the series converges if $\rho < 1$,

(b) the series diverges if $\rho > 1$ or ρ is infinite, and

(c) the test is inconclusive if $\rho = 1$.

Note. Again, the Root Test doesn't require a second series and is easier to use than the Direct Comparison Test.

Example. Number 34 page 649.