Chapter 1. Vectors, Matrices, and Linear Systems
Section 1.1. Vectors in Euclidean Spaces—Proofs of Theorems
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Page 16 Number 10

Page 16 Number 10. Compute the linear combination $3\vec{u} + \vec{v} - \vec{w}$ where \( \vec{u} = [1, 2, 1, 0], \vec{v} = [-2, 0, 1, 6], \) and \( \vec{w} = [3, -5, 1, -2]. \)

Solution. We have

\[
3\vec{u} + \vec{v} - \vec{w} = 3[1, 2, 1, 0] + [-2, 0, 1, 6] - [3, -5, 1, -2] \\
= [3(1), 3(2), 3(1), 3(0)] + [-2, 0, 1, 6] - [3, -5, 1, -2] \\
\text{by Definition 1.1(3), "Scalar Multiplication"} \\
= [3, 6, 3, 0] + [-2, 0, 1, 6] - [3, -5, 1, -2] \text{ simplifying}
\]
Page 16 Number 10

Compute the linear combination $3\vec{u} + \vec{v} - \vec{w}$ where $
\vec{u} = [1, 2, 1, 0], \ \vec{v} = [-2, 0, 1, 6], \text{ and } \vec{w} = [3, -5, 1, -2].$

Solution. We have

\[
3\vec{u} + \vec{v} - \vec{w} = 3[1, 2, 1, 0] + [-2, 0, 1, 6] - [3, -5, 1, -2]
\]

\[
= [3(1), 3(2), 3(1), 3(0)] + [-2, 0, 1, 6] - [3, -5, 1, -2]
\]

by Definition 1.1(3), “Scalar Multiplication”

\[
= [3, 6, 3, 0] + [-2, 0, 1, 6] - [3, -5, 1, -2]
\]

simplifying

\[
= [3 + (-2), 6 + 0, 3 + 1, 0 + 6] - [3, -5, 1, -2]
\]

by Definition 1.1(1), “Vector Addition”

\[
= [1, 6, 4, 6] - [3, -5, 1, -2]
\]

simplifying
Compute the linear combination $3\vec{u} + \vec{v} - \vec{w}$ where $\vec{u} = [1, 2, 1, 0]$, $\vec{v} = [-2, 0, 1, 6]$, and $\vec{w} = [3, -5, 1, -2]$.

**Solution.** We have

$$3\vec{u} + \vec{v} - \vec{w} = 3[1, 2, 1, 0] + [-2, 0, 1, 6] - [3, -5, 1, -2]$$

by Definition 1.1(3), “Scalar Multiplication”

$$= [3(1), 3(2), 3(1), 3(0)] + [-2, 0, 1, 6] - [3, -5, 1, -2]$$

simplifying

$$= [3 + (-2), 6 + 0, 3 + 1, 0 + 6] - [3, -5, 1, -2]$$

by Definition 1.1(1), “Vector Addition”

$$= [1, 6, 4, 6] - [3, -5, 1, -2]$$

simplifying

$$= [1 - (3), 6 - (-5), 4 - (1), 6 - (-2)]$$

by Definition 1.1(2), “Vector Subtraction”

$$= [-2, 11, 3, 8]$$

simplifying.

So we conclude $3\vec{u} + \vec{v} - \vec{w} = [-2, 11, 3, 8]$. □
Compute the linear combination $3\vec{u} + \vec{v} - \vec{w}$ where $\vec{u} = [1, 2, 1, 0]$, $\vec{v} = [-2, 0, 1, 6]$, and $\vec{w} = [3, -5, 1, -2]$.

**Solution.** We have

$$3\vec{u} + \vec{v} - \vec{w} = 3[1, 2, 1, 0] + [-2, 0, 1, 6] - [3, -5, 1, -2]$$

by Definition 1.1(3), "Scalar Multiplication"

$$= [3(1), 3(2), 3(1), 3(0)] + [-2, 0, 1, 6] - [3, -5, 1, -2]$$

simplifying

$$= [3 + (-2), 6 + 0, 3 + 1, 0 + 6] - [3, -5, 1, -2]$$

by Definition 1.1(1), "Vector Addition"

$$= [1 + 6, 4, 4 + 1, 6] - [3, -5, 1, -2]$$

simplifying

$$= [1 - (3), 6 - (-5), 4 - (1), 6 - (-2)]$$

by Definition 1.1(2), "Vector Subtraction"

$$= [-2, 11, 3, 8]$$

So we conclude $3\vec{u} + \vec{v} - \vec{w} = [-2, 11, 3, 8]$. □
Page 16 Number 14. Reproduce the vectors in this figure and draw an arrow representing $-3\vec{u} + 2\vec{w}$.

Solution. From Definition 1.1(3), “Scalar Multiplication,” and the geometric interpretation of vectors (see the class notes, pages 2, 3, and 4) we represent $-3\vec{u}$ and $2\vec{w}$ as:
Page 16 Number 14. Reproduce the vectors in this figure and draw an arrow representing $-3\vec{u} + 2\vec{w}$.

**Solution.** From Definition 1.1(3), “Scalar Multiplication,” and the geometric interpretation of vectors (see the class notes, pages 2, 3, and 4) we represent $-3\vec{u}$ and $2\vec{w}$ as:
Page 16 Number 14. Reproduce the vectors in this figure and draw an arrow representing $-3\vec{u} + 2\vec{w}$.

Solution. From Definition 1.1(3), “Scalar Multiplication,” and the geometric interpretation of vectors (see the class notes, pages 2, 3, and 4) we represent $-3\vec{u}$ and $2\vec{w}$ as:
Then by the parallelogram property of addition:
Let \( \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n \) and let \( r, s \) be scalars in \( \mathbb{R} \).

Prove (A1): \((\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})\).

**Proof.** Since \( \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n \), by Definition 1.A, “Vectors in \( \mathbb{R}^n \),” we have that
\[
\vec{u} = [u_1, u_2, \ldots, u_n], \quad \vec{v} = [v_1, v_2, \ldots, v_n], \quad \text{and} \quad \vec{w} = [w_1, w_2, \ldots, w_n],
\]
where all \( u_i, v_i, \) and \( w_i \) are real numbers.
Page 17 Number 40(a). Let \( \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n \) and let \( r, s \) be scalars in \( \mathbb{R} \). Prove \((A1)\): \((\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})\).

**Proof.** Since \( \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n \), by Definition 1.A, “Vectors in \( \mathbb{R}^n \),” we have that \( \vec{u} = [u_1, u_2, \ldots, u_n] \), \( \vec{v} = [v_1, v_2, \ldots, v_n] \), and \( \vec{w} = [w_1, w_2, \ldots, w_n] \) where all \( u_i, v_i, \) and \( w_i \) are real numbers. Then

\[
(\vec{u} + \vec{v}) + \vec{w} = ([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) + [w_1, w_2, \ldots, w_n]
\]

\[
= [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] + [w_1, w_2, \ldots, w_n]
\]

by Definition 1.1(1), “Vector Addition”
Let \( \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n \) and let \( r, s \) be scalars in \( \mathbb{R} \).

Prove (A1): \((\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})\).

**Proof.** Since \( \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n \), by Definition 1.A, “Vectors in \( \mathbb{R}^n \),” we have that \( \vec{u} = [u_1, u_2, \ldots, u_n] \), \( \vec{v} = [v_1, v_2, \ldots, v_n] \), and \( \vec{w} = [w_1, w_2, \ldots, w_n] \) where all \( u_i, v_i, \) and \( w_i \) are real numbers. Then

\[
(\vec{u} + \vec{v}) + \vec{w} = ([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) + [w_1, w_2, \ldots, w_n]
\]

\[
= [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] + [w_1, w_2, \ldots, w_n]
\]

by Definition 1.1(1), “Vector Addition”

\[
= [(u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \ldots, (u_n + v_n) + w_n]
\]

by Definition 1.1(1), “Vector Addition”
Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and let $r, s$ be scalars in $\mathbb{R}$. Prove (A1): $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.

**Proof.** Since $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, by Definition 1.1A, “Vectors in $\mathbb{R}^n$,” we have that $\vec{u} = [u_1, u_2, \ldots, u_n]$, $\vec{v} = [v_1, v_2, \ldots, v_n]$, and $\vec{w} = [w_1, w_2, \ldots, w_n]$ where all $u_i$, $v_i$, and $w_i$ are real numbers. Then

$$
(\vec{u} + \vec{v}) + \vec{w} = ([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) + [w_1, w_2, \ldots, w_n]
$$

$$
= [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] + [w_1, w_2, \ldots, w_n]
$$

by Definition 1.1(1), “Vector Addition”

$$
= [(u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \ldots, (u_n + v_n) + w_n]
$$

by Definition 1.1(1), “Vector Addition”

$$
= [u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n)]
$$

since addition of real numbers is associative.
Page 17 Number 40(a). Let \( \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n \) and let \( r, s \) be scalars in \( \mathbb{R} \).

Prove (A1): \((\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})\).

Proof. Since \( \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n \), by Definition 1.1A, "Vectors in \( \mathbb{R}^n \)," we have that

\[
\vec{u} = [u_1, u_2, \ldots, u_n], \quad \vec{v} = [v_1, v_2, \ldots, v_n], \quad \text{and} \quad \vec{w} = [w_1, w_2, \ldots, w_n]
\]

where all \( u_i, v_i, \) and \( w_i \) are real numbers. Then

\[
(\vec{u} + \vec{v}) + \vec{w} = ([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) + [w_1, w_2, \ldots, w_n] \\
= [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] + [w_1, w_2, \ldots, w_n] \\
\text{by Definition 1.1(1), "Vector Addition"} \\
= [(u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \ldots, (u_n + v_n) + w_n] \\
\text{by Definition 1.1(1), "Vector Addition"} \\
= [u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n)]
\]

since addition of real numbers is associative.
Page 17 Number 40(a). Let \( \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n \) and let \( r, s \) be scalars in \( \mathbb{R} \). Prove (A1): \((\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})\).

Proof (continued). . . .

\[
(\vec{u} + \vec{v}) + \vec{w} = \left[ u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n) \right] \\
= \left[ u_1, u_2, \ldots, u_n \right] + \left[ v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n \right] \\
\text{by Definition 1.1(1), “Vector Addition”}
\]
Let \( \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \) and let \( r, s \) be scalars in \( \mathbb{R} \).

Prove (A1): \((\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})\).

**Proof (continued).**

\[
(\mathbf{u} + \mathbf{v}) + \mathbf{w} = [u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n)]
\]
\[
= [u_1, u_2, \ldots, u_n] + [v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n]
\]
by Definition 1.1(1), “Vector Addition”
\[
= [u_1, u_2, \ldots, u_n] + ([v_1, v_2, \ldots, v_n] + [w_1, w_2, \ldots, w_n])
\]
by Definition 1.1(1), “Vector Addition”
\[
= \mathbf{u} + (\mathbf{v} + \mathbf{w}).
\]
Page 17 Number 40(a). Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and let $r, s$ be scalars in $\mathbb{R}$.
Prove (A1): $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.

Proof (continued). . . .

$$\begin{align*}
(\vec{u} + \vec{v}) + \vec{w} & = [u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n)] \\
& = [u_1, u_2, \ldots, u_n] + [v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n] \\
& \text{by Definition 1.1(1), "Vector Addition"} \\
& = [u_1, u_2, \ldots, u_n] + ([v_1, v_2, \ldots, v_n] + [w_1, w_2, \ldots, w_n]) \\
& \text{by Definition 1.1(1), "Vector Addition"} \\
& = \vec{u} + (\vec{v} + \vec{w}).
\end{align*}$$
Page 17 Number 41(a). Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ and let $r$ be a scalar in $\mathbb{R}$. Prove (S1): $r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$.

**Proof.** Since $\vec{v}, \vec{w} \in \mathbb{R}^n$, by Definition 1.A, “Vectors in $\mathbb{R}^n$,” we have that $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$ where all $v_i$ and $w_i$ are real numbers.
Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ and let $r$ be a scalar in $\mathbb{R}$. Prove (S1): $r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$.

**Proof.** Since $\vec{v}, \vec{w} \in \mathbb{R}^n$, by Definition 1.A, “Vectors in $\mathbb{R}^n$,” we have that $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$ where all $v_i$ and $w_i$ are real numbers. Then

\[
\begin{align*}
    r(\vec{v} + \vec{w}) &= r([v_1, v_2, \ldots, v_n] + [w_1, w_2, \ldots, w_n]) \\
    &= r[v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n] \\
    &\quad \text{by Definition 1.1(1), “Vector Addition”}
\end{align*}
\]
Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ and let $r$ be a scalar in $\mathbb{R}$. Prove (S1): $r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$.

**Proof.** Since $\vec{v}, \vec{w} \in \mathbb{R}^n$, by Definition 1.A, “Vectors in $\mathbb{R}^n$,” we have that $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$ where all $v_i$ and $w_i$ are real numbers. Then

$$r(\vec{v} + \vec{w}) = r([v_1, v_2, \ldots, v_n] + [w_1, w_2, \ldots, w_n])$$

$$= r[v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n]$$

by Definition 1.1(1), “Vector Addition”

$$= [r(v_1 + w_1), r(v_2 + w_2), \ldots, r(v_n + w_n)]$$

by Definition 1.1(3), “Scalar Multiplication”
Page 17 Number 41(a). Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ and let $r$ be a scalar in $\mathbb{R}$. Prove (S1): $r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$.

Proof. Since $\vec{v}, \vec{w} \in \mathbb{R}^n$, by Definition 1.1A, “Vectors in $\mathbb{R}^n$,” we have that $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$ where all $v_i$ and $w_i$ are real numbers. Then

\[
\begin{align*}
    r(\vec{v} + \vec{w}) &= r([v_1, v_2, \ldots, v_n] + [w_1, w_2, \ldots, w_n]) \\
    &= r[v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n] \\
    &= [r(v_1 + w_1), r(v_2 + w_2), \ldots, r(v_n + w_n)] \\
    &= [rv_1 + rw_1, rv_2 + rw_2, \ldots, rv_n + rw_n]
\end{align*}
\]

by Definition 1.1(1), “Vector Addition”

by Definition 1.1(3), “Scalar Multiplication”

since multiplication distributes over addition in the real numbers...
Let \( \vec{v}, \vec{w} \in \mathbb{R}^n \) and let \( r \) be a scalar in \( \mathbb{R} \). Prove (S1): 
\[ r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}. \]

**Proof.** Since \( \vec{v}, \vec{w} \in \mathbb{R}^n \), by Definition 1.A, “Vectors in \( \mathbb{R}^n \),” we have that \( \vec{v} = [v_1, v_2, \ldots, v_n] \) and \( \vec{w} = [w_1, w_2, \ldots, w_n] \) where all \( v_i \) and \( w_i \) are real numbers. Then

\[
\begin{align*}
    r(\vec{v} + \vec{w}) &= r([v_1, v_2, \ldots, v_n] + [w_1, w_2, \ldots, w_n]) \\
    &= r[v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n] \\
    & \quad \text{by Definition 1.1(1), “Vector Addition”} \\
    &= [r(v_1 + w_1), r(v_2 + w_2), \ldots, r(v_n + w_n)] \\
    & \quad \text{by Definition 1.1(3), “Scalar Multiplication”} \\
    &= [rv_1 + rw_1, rv_2 + rw_2, \ldots, rv_n + rw_n] \\
    & \quad \text{since multiplication distributes} \\
    & \quad \text{over addition in the real numbers...}
\end{align*}
\]
Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ and let $r$ be a scalar in $\mathbb{R}$. Prove (S1): $r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$.

Proof (continued). . .

\[
\begin{align*}
r(\vec{v} + \vec{w}) &= [rv_1 + rw_1, rv_2 + rw_2, \ldots, rv_n + rw_n] \\
&= [rv_1, rv_2, \ldots, rv_n] + [rw_1, rw_2, \ldots, rw_n] \\
&\quad \text{by Definition 1.1(1), “Vector Addition”} \\
&= r[v_1, v_2, \ldots, v_n] + r[w_1, w_2, \ldots, w_n] \\
&\quad \text{by Definition 1.1(3), “Scalar Multiplication”} \\
&= r\vec{v} + r\vec{w}.
\end{align*}
\]
Page 17 Number 41(a). Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ and let $r$ be a scalar in $\mathbb{R}$. Prove (S1): $r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$.

Proof (continued). . .


d_r(\vec{v} + \vec{w}) = \begin{bmatrix} rv_1 + rw_1, rv_2 + rw_2, \ldots, rv_n + rw_n \end{bmatrix} \\
= \begin{bmatrix} rv_1, rv_2, \ldots, rv_n \end{bmatrix} + \begin{bmatrix} rw_1, rw_2, \ldots, rw_n \end{bmatrix} \\
by \text{Definition 1.1(1), "Vector Addition"}
= r\begin{bmatrix} v_1, v_2, \ldots, v_n \end{bmatrix} + r\begin{bmatrix} w_1, w_2, \ldots, w_n \end{bmatrix} \\
by \text{Definition 1.1(3), "Scalar Multiplication"}
= r\vec{v} + r\vec{w}.
Find all scalars $c$ (if any) such that the vector $[c^2, -4]$ is parallel to the vector $[1, -2]$.

**Solution.** By Definition 1.2, two nonzero vectors are parallel if one is a scalar multiple of the other, say $[c^2, -4] = r[1, -2]$ for scalar $r \in \mathbb{R}$. Then by Definition 1.1(3), “Scalar Multiplication,” $[c^2, -4] = [r, -2r]$. 
Page 16 Number 22. Find all scalars \( c \) (if any) such that the vector \([c^2, -4]\) is parallel to the vector \([1, -2]\).

**Solution.** By Definition 1.2, two nonzero vectors are parallel if one is a scalar multiple of the other, say \([c^2, -4] = r [1, -2]\) for scalar \( r \in \mathbb{R} \). Then by Definition 1.1(3), “Scalar Multiplication,” \([c^2, -4] = [r, -2r]\). So we need both \( c^2 = r \) and \(-4 = -2r\). Since \(-4 = -2r\) then we must have \( r = 2 \). With \( r = 2 \) and \( c^2 = r = 2 \) we must have that either \( c = \sqrt{2} \) or \( c = -\sqrt{2} \). □
**Page 16 Number 22.** Find all scalars \( c \) (if any) such that the vector \([c^2, -4]\) is parallel to the vector \([1, -2]\).

**Solution.** By Definition 1.2, two nonzero vectors are parallel if one is a scalar multiple of the other, say \([c^2, -4] = r[1, -2]\) for scalar \( r \in \mathbb{R} \). Then by Definition 1.1(3), “Scalar Multiplication,” \([c^2, -4] = [r, -2r]\). So we need both \( c^2 = r \) and \(-4 = -2r\). Since \(-4 = -2r\) then we must have \( r = 2\). With \( r = 2 \) and \( c^2 = r = 2 \) we must have that either \( c = \sqrt{2} \) or \( c = -\sqrt{2} \). □
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**Page 16 Number 28.** Find all scalars $c$ (if any) such that the vector $\vec{v} + c\vec{j} + (c - 1)\vec{k}$ is in the span of $\vec{i} + 2\vec{j} + \vec{k}$ and $3\vec{i} + 6\vec{j} + 3\vec{k}$.

**Solution.** By Definition 1.4, the span of $\vec{i} + 2\vec{j} + \vec{k}$ and $3\vec{i} + 6\vec{j} + 3\vec{k}$ is the set of all linear combinations of these two vectors. So the question becomes: For which $c \in \mathbb{R}$ is

$$\vec{v} + c\vec{j} + (c - 1)\vec{k} = r_1(\vec{i} + 2\vec{j} + \vec{k}) + r_2(3\vec{i} + 6\vec{j} + 3\vec{k})$$

for some $r_1, r_2 \in \mathbb{R}$?
Find all scalars $c$ (if any) such that the vector $\vec{i} + c\vec{j} + (c - 1)\vec{k}$ is in the span of $\vec{i} + 2\vec{j} + \vec{k}$ and $3\vec{i} + 6\vec{j} + 3\vec{k}$.

**Solution.** By Definition 1.4, the span of $\vec{i} + 2\vec{j} + \vec{k}$ and $3\vec{i} + 6\vec{j} + 3\vec{k}$ is the set of all linear combinations of these two vectors. So the question becomes: For which $c \in \mathbb{R}$ is

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for some $r_1, r_2 \in \mathbb{R}$? If this holds, $\vec{i} + c\vec{j} + (c - 1)\vec{k} = (r_1 + 3r_2)\vec{i} + (2r_1 + 6r_2)\vec{j} + (r_1 + 3r_2)\vec{k}$.

So we need $c \in \mathbb{R}$ such that

$$1 = r_1 + 3r_2 \quad (1)$$

$$c = 2r_1 + 6r_2 \quad (2)$$

$$c - 1 = r_1 + 3r_2 \quad (3)$$

Multiplying (1) by 2 gives $2 = 2r_1 + 6r_2$. Combining this with (2) we see that we need $c = 2$. With $c = 2$, equation (3) gives $1 = r_1 + 3r_2$ which is (1). Therefore all three equations (1), (2), and (3) are satisfied when $c = 2$. We can take $r_1 = 1$ and $r_2 = 0$, for example. □
**Page 16 Number 28.** Find all scalars \( c \) (if any) such that the vector \( \vec{i} + c\vec{j} + (c - 1)\vec{k} \) is in the span of \( \vec{i} + 2\vec{j} + \vec{k} \) and \( 3\vec{i} + 6\vec{j} + 3\vec{k} \).

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Solution. By Definition 1.4, the span of $\vec{i} + 2\vec{j} + \vec{k}$ and $3\vec{i} + 6\vec{j} + 3\vec{k}$ is the set of all linear combinations of these two vectors. So the question becomes: For which $c \in \mathbb{R}$ is $\vec{i} + c\vec{j} + (c - 1)\vec{k} = r_1(\vec{i} + 2\vec{j} + \vec{k}) + r_2(3\vec{i} + 6\vec{j} + 3\vec{k})$ for some $r_1, r_2 \in \mathbb{R}$? If this holds, $\vec{i} + c\vec{j} + (c - 1)\vec{k} = (r_1 + 3r_2)\vec{i} + (2r_1 + 6r_2)\vec{j} + (r_1 + 3r_2)\vec{k}$.

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