Example 1.4.A

Example 1.4.A. Solve the system

\[ \begin{align*}
  x_1 + 3x_2 - x_3 &= 4 \\
  x_2 - x_3 &= -1 \\
  x_3 &= 3.
\end{align*} \tag{1, 2, 3} \]

Solution. From (3) we have \( x_3 = 3 \). So from (2) we have \( x_2 - (3) = -1 \) or \( x_2 = 2 \). Then from (1) we have \( x_1 + 3(2) - (3) = 4 \) or \( x_1 = 1 \). So the solution is \( x_1 = 1, x_2 = 2, x_3 = 3 \). \( \square \)

Example 1.4.B

Example 1.4.B. Is this system consistent or inconsistent:

\[ \begin{align*}
  2x_1 + x_2 - x_3 &= 1 \\
  x_1 - x_2 + 3x_3 &= 1 \\
  3x_1 + 2x_3 &= 3.
\end{align*} \]

Solution. We create the augmented matrix for the system:

\[ \begin{bmatrix}
  2 & 1 & -1 & 1 \\
  1 & -1 & 3 & 1 \\
  3 & 0 & 2 & 3
\end{bmatrix}. \]

We now use elementary row operations to put the augmented matrix in row-echelon form. We have

\[ \begin{bmatrix}
  2 & 1 & -1 & 1 \\
  1 & -1 & 3 & 1 \\
  3 & 0 & 2 & 3
\end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\rightarrow} \begin{bmatrix}
  1 & -1 & 3 & 1 \\
  2 & 1 & -1 & 1 \\
  3 & 0 & 2 & 3
\end{bmatrix} \]

\[ \rightarrow \begin{bmatrix}
  1 & -1 & 3 & 1 \\
  0 & 3 & -7 & 1 \\
  0 & 3 & -7 & 1
\end{bmatrix}. \]

Solution (continued).

\[ \begin{align*}
  &R_2 \rightarrow R_2 - 2R_1 \\
  &R_3 \rightarrow R_3 - 3R_1 \\
  &R_3 \rightarrow R_3 - R_2
\end{align*} \]

\[ \begin{bmatrix}
  1 & -1 & 3 & 1 \\
  0 & 3 & -7 & 1 \\
  0 & 3 & -7 & 1
\end{bmatrix} \overset{\text{(*)}}{\rightarrow} \begin{bmatrix}
  1 & -1 & 3 & 1 \\
  0 & 3 & -7 & 1 \\
  0 & 0 & 0 & 0
\end{bmatrix}. \]

\[ \overset{\text{(**)}}{\rightarrow} \begin{bmatrix}
  1 & -1 & 3 & 1 \\
  0 & 3 & -7 & 1 \\
  0 & 0 & 1 & 1
\end{bmatrix}. \]
Example 1.4.B (continued 2)

**Example 1.4.B.** Is this system consistent or inconsistent:

\[
\begin{align*}
2x_1 + x_2 - x_3 &= 1 \\
x_1 - x_2 + 3x_3 &= 1 \\
3x_1 + 2x_3 &= 3?
\end{align*}
\]

**Solution (continued).** Now by Theorem 1.6 (Invariance of Solution Sets) we see that the original system has the same solution (if one exists) as each of the systems associated with any of these augmented matrices. We see that we have a problem in (*) since the second and third rows imply that \(3x_2 - 7x_3 = -1\) and \(3x_2 - 7x_3 = 0\). Of course, both of these cannot be true so this tells us that there is no solution. Alternatively, the augmented matrix in (**) is in row-echelon form (by Definition 1.12) and the third row of (**) implies that \(0 = 1\), which of course is not the case and so the original system has no solution and is [inconsistent. □

Example 1.4.C (continued 1)

**Example 1.4.C.** Is this system consistent or inconsistent:

\[
\begin{align*}
2x_1 + x_2 - x_3 &= 1 \\
x_1 - x_2 + 3x_3 &= 1 \\
3x_1 + 2x_3 &= 2?
\end{align*}
\]

(HINT: This system has multiple solutions. Express the solutions in terms of an unknown parameter \(r\)).

**Solution.** We take the same approach as in the previous example. The augmented matrix is similar to the one in the previous example and we perform the same row operations (so we give less arithmetic details). So

\[
\begin{bmatrix}
2 & 1 & -1 & 1 \\
1 & -1 & 3 & 1 \\
3 & 0 & 2 & 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -1 & 3 & 1 \\
0 & 3 & -7 & -1 \\
0 & 3 & -7 & -1
\end{bmatrix}
\]

**Example 1.4.C (continued 2)

**Solution (continued).** We introduce a parameter \(r\) for \(x_3\) where \(r\) can be any real number. We then have

\[
\begin{align*}
x_1 &= 2/3 - (2/3)r \\
x_2 &= -1/3 + (7/3)r \\
x_3 &= r.
\end{align*}
\]

We can check to see that for any \(r \in \mathbb{R}\), this is a solution to each of the original equations:

\[
\begin{align*}
2x_1 + x_2 - x_3 &= 2(2/3 - (2/3)r) + (-1/3 + (7/3)r) - (r) = 1 \\
x_1 - x_2 + 3x_3 &= (2/3 - (2/3)r) - (-1/3 + (7/3)r) + 3(r) = 1 \\
3x_1 + 2x_3 &= 3(2/3 - (2/3)r) + 2(r) = 2.
\end{align*}
\]

So the original system is [consistent] and for any \(r \in \mathbb{R}\) a solution is given by \(x_1 = 2/3 - (2/3)r, x_2 = -1/3 + (7/3)r, x_3 = r\). Expressed as a vector equation, we have \(\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ 0 \end{bmatrix} + r \begin{bmatrix} -2/3 \\ 7/3 \\ 1 \end{bmatrix}\). □
Page 68 Number 2(a). Use elementary row operations to put
\[
\begin{bmatrix}
2 & 4 & -2 \\
4 & 8 & 3 \\
-1 & -3 & 0
\end{bmatrix}
\]
in row-echelon form (REF).

Solution. To make the arithmetic easier, we move the -1 in position (3,1) to position (1,1) using the row operation \( R_1 \leftrightarrow R_3 \); we then get zeros below the pivot at position (1,1) using Step (2b) of the previous note. Then we’ll deal with the second column. So
\[
\begin{bmatrix}
2 & 4 & -2 \\
4 & 8 & 3 \\
-1 & -3 & 0
\end{bmatrix}
\begin{bmatrix}
R_1 \leftrightarrow R_3 \\
R_2 \rightarrow R_2 + 4R_1 \\
R_3 \rightarrow R_3 + 2R_1
\end{bmatrix}
\begin{bmatrix}
-1 & -3 & 0 \\
4 + 4(-1) & 8 + 4(-3) & 3 + 4(0) \\
2 + 2(-1) & 4 + 2(-3) & -2 + 2(0)
\end{bmatrix}
\]
\[
\begin{bmatrix}
-1 & -3 & 0 \\
0 & -2 & -2 \\
0 & 0 & 7
\end{bmatrix}
\]
This matrix satisfies Definition 1.12 and so is in REF. A word of warning: A row-echelon form of a matrix is not unique! For example, we could perform the elementary row operations \( R_1 \rightarrow -R_1, R_2 \rightarrow R_2 /(-2), \) and \( R_3 \rightarrow R_3/7 \) (to make all pivots 1) and get
\[
\begin{bmatrix}
1 & 3 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]
as an alternative row echelon form of the given matrix.

Page 69 Number 16(a). Consider
\[
\begin{align*}
2x + y - 3z &= 0 \\
6x + 3y - 8z &= 0 \\
2x - y + 5z &= -4.
\end{align*}
\]
Put the augmented matrix in row-echelon form and use back substitution to solve.

Solution. We have
\[
\begin{bmatrix}
2 & 1 & -3 & 0 \\
6 & 3 & -8 & 0 \\
2 & -1 & 5 & -4
\end{bmatrix}
\begin{bmatrix}
R_2 \rightarrow R_2 - 3R_1 \\
R_3 \rightarrow R_3 - R_1
\end{bmatrix}
\begin{bmatrix}
2 & 1 & -3 & 0 \\
6 - 3(2) & 3 - 3(1) & -8 - 3(-3) & 0 - 3(0) \\
2 - (2) & -1 - (1) & 5 - (-3) & -4 - (0)
\end{bmatrix}
\]
So this matrix is in row echelon form (by Definition 1.12) and the associated system of equations for this matrix is
\[
\begin{align*}
2x + y - 3z &= 0 \quad (1) \\
-2y + 8z &= -4 \quad (2) \\
z &= 0 \quad (3)
\end{align*}
\]
By back substitution, (3) gives \( z = 0. \) Then (2) gives \(-2y + 8(0) = -4 \) or \( y = 2. \) From (1) we have \( 2x + (2) - 3(0) = 0 \) or \( x = -1. \) So the (unique) solution to the system of equations is \([x = -1, y = 2, z = 0].\)
**Page 68 Number 2(b).** Use elementary row operations to put
\[
\begin{bmatrix}
2 & 4 & -2 \\
4 & 8 & 3 \\
-1 & -3 & 0
\end{bmatrix}
\] in reduced row-echelon form (RREF).

**Solution.** In Number 2(a) we saw that
\[
\begin{bmatrix}
2 & 4 & -2 \\
4 & 8 & 3 \\
-1 & -3 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 3 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}.
\]

Since the pivots are all 1 in this new REF matrix, we only need to apply elementary row operations to get 0’s above the pivots. We have
\[
\begin{bmatrix}
2 & 4 & -2 \\
4 & 8 & 3 \\
-1 & -3 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 3 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

**Page 69 Number 16(b). Consider**

\[
\begin{align*}
2x + y - 3z &= 0 \\
6x + 3y - 8z &= 0 \\
2x - y + 5z &= -4.
\end{align*}
\]

Put the augmented matrix in reduced row-echelon form and solve.

**Solution.** In Number 16(a) we saw that the augmented matrix for this system is
\[
\begin{bmatrix}
2 & 1 & -3 & 0 \\
6 & 3 & -8 & 0 \\
2 & -1 & 5 & -4
\end{bmatrix} \sim \begin{bmatrix}
2 & 1 & -3 & 0 \\
0 & -2 & 8 & -4 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

So we need to continue with elementary row operations to make all pivots 1 and make all entries above pivots 0.

---

**Page 68 Number 2(b) (continued)**

**Solution (continued).**

\[
\begin{align*}
R_1 \rightarrow R_1 - 3R_2 & \quad \begin{bmatrix}
1 - 3(0) & 3 - 3(1) & 0 - 3(1) \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & -3 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}

R_1 \rightarrow R_1 + 3R_3 & \quad \begin{bmatrix}
1 + 3(0) & 0 + 3(0) & -3 + 3(1) \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}

R_2 \rightarrow R_2 - R_3 & \quad \begin{bmatrix}
2 - (0) & 1 - (1) & -3 - (-4) \\
0 & 1 & -4 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
2 & 0 & 1 & -2 \\
0 & 1 & -4 & 2 \\
0 & 0 & 1 & 0
\end{bmatrix}

R_1 \rightarrow R_1 - R_2 & \quad \begin{bmatrix}
2 - (0) & 0 - (0) & 1 - (1) \\
0 & 0 & 1
\end{bmatrix}
\]

This is in reduced row echelon form by the previous definition. Notice that requiring the pivots to all be 1 in a RREF matrix guarantees that there is a unique RREF matrix which is row equivalent to any given matrix. □
Solution (continued).

\[
\begin{bmatrix}
2 & 0 & 0 & -2 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\xrightarrow{R_1\rightarrow R_1/2}
\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}.
\]

So associated with this augmented matrix is a system of equations that allows us to just read off the solution: \(x = -1, y = 2, z = 0\). By Theorem 1.16 (Invariance of Solution Sets) this is the solution to the original system of equations. Notice that this is the same solution as obtained in Number 16(a), though we have avoided back substitution by performing more elementary row operations.  

Theorem 1.7 (continued 1)

Proof (continued). No values of \(x_1, x_2, \ldots, x_n\) make
\[0x_1 + 0x_2 + \cdots + 0x_n = c_i \neq 0,\]
so the solution set of \(H\bar{x} = \bar{c}\) is empty and hence the solution set of \(A\bar{x} = \bar{b}\) is empty. That is, \(A\bar{x} = \bar{b}\) has no solution and so is inconsistent.

Now suppose that \([H \mid \bar{c}]\) has no row with all entries 0 to the left of the partition and a nonzero entry to the right. We’ll show that in this case \(A\bar{x} = \bar{b}\) has a solution and so is consistent, completing the proof of part (1). If a row of \([H \mid \bar{c}]\) is all zeros on both sides of the partition, then this corresponds to the equation \(0x_1 + 0x_2 + \cdots + 0x_n = 0\) which is satisfied by all \(x_1, x_2, \ldots, x_n\) and so this equation contributes no information in determining a solution to \(A\bar{x} = \bar{b}\). So we can create matrix \([H' \mid \bar{c}]\) by eliminating all rows of 0’s in matrix \([H \mid \bar{c}]\). So every row of \(H'\) contains a pivot.

Theorem 1.7 (continued 2)

Proof (continued). If every column of \(H'\) (and hence of \(H\)) contains a pivot then each \(x_j\) is uniquely determined and so the system is consistent and has a unique solution and (2) follows. If \(A\bar{x} = \bar{b}\) is consistent and the \(j\)th column of \(H'\) (and hence of \(H\)) contains no pivot then \(x_j\) can take on any value (it is a free variable, as in Example 1.4.C). The other \(x_i\), where column \(i\) contains a pivot, can then be determined in terms of these free variables and so (3) holds.

Notice that when (2) holds there is a unique solution and when (3) holds there are multiple solutions. In either case, the system is consistent and now (1) follows.
**Page 71 Number 52. Proof for Row Interchange (Theorem 1.8).**

Suppose $E$ results from interchanging Row $i$ and Row $j$ in $\mathcal{I}$: $\mathcal{I} \xrightarrow{R_i \leftrightarrow R_j} E$.

Then the $k$th row of $E$ is $[0, 0, \ldots, 0, 1, 0, \ldots, 0]$, where

1. for $k \not\in \{i, j\}$ the nonzero entry is the $k$th entry,
2. for $k = i$ the nonzero entry is the $j$th entry, and
3. for $k = j$ the nonzero entry is the $i$th entry.

Let $A = [a_{ij}]$, $E = [e_{ij}]$, and $B = [b_{ij}] = EA$. The $k$th row of $B$ is $[b_{k1}, b_{k2}, \ldots, b_{kn}]$ and

$$b_{k\ell} = \sum_{p=1}^{n} e_{kp} a_{p\ell}.$$

Now if $k \not\in \{i, j\}$ then all $e_{kp}$ are 0 except for $p = k$ and

$$b_{k\ell} = \sum_{p=1}^{n} e_{kp} a_{p\ell} = e_{kk} a_{k\ell} = (1) a_{k\ell} = a_{k\ell}.$$

**Proof (continued).** Therefore for $k \not\in \{i, j\}$, the $k$th row of $B$ is the same as the $k$th row of $A$. If $k = i$ then all $e_{kp}$ are 0 except for $p = j$ and

$$b_{k\ell} = b_{i\ell} = \sum_{p=1}^{n} e_{kp} a_{p\ell} = e_{kj} a_{j\ell} = (1) a_{j\ell} = a_{j\ell}$$

and the $i$th row of $B$ is the same as the $j$th row of $A$. Similarly, if $k = j$ then all $e_{kp}$ are 0 except for $p = i$ and

$$b_{k\ell} = b_{j\ell} = \sum_{p=1}^{n} e_{kp} a_{p\ell} = e_{ki} a_{i\ell} = (1) a_{i\ell} = a_{i\ell}$$

and the $j$th row of $B$ is the same as the $i$th row of $A$. Therefore

$$B = EA$$

and $A \xrightarrow{R_i \leftrightarrow R_j} B$.

---

**Page 71 Number 54. Proof for Row Addition (Theorem 1.8).**

Suppose $E$ results from adding $s$ times Row $j$ to Row $i$ in $\mathcal{I}$: $\mathcal{I} \xrightarrow{R_i \rightarrow \mathcal{I} + sR_j} E$. Then the $k$th row of $E$ is the same as the $k$th row of $\mathcal{I}$ for $k \neq i$, and the $i$th row of $E$ is $[0, 0, \ldots, 0, 1, 0, \ldots, 0, s, 0, \ldots, 0, 0]$ (or $[0, 0, \ldots, 0, s, 0, \ldots, 0, 1, 0, \ldots, 0, 0]$) where the $i$th component is 1 and the $j$th component is $s$ and all other components are 0. Let $A = [a_{ij}]$, $E = [e_{ij}]$, and $B = [b_{ij}] = EA$. The $k$th row of $B$ is $[b_{k1}, b_{k2}, \ldots, b_{kn}]$ and

$$b_{k\ell} = \sum_{p=1}^{n} e_{kp} a_{p\ell}.$$

For $k \neq i$,

$$b_{k\ell} = \sum_{p=1}^{n} e_{kp} a_{p\ell} = e_{kk} a_{k\ell} = (1) a_{k\ell} = a_{k\ell}.$$

Therefore for $k \neq i$, the $k$th row of $B$ is the same as the $k$th row of $A$.

**Proof (continued).** If $k = i$

$$b_{k\ell} = b_{i\ell} = \sum_{p=1}^{n} e_{kp} a_{p\ell} = e_{ii} a_{i\ell} + e_{ij} a_{j\ell} = (1) a_{i\ell} + (s) a_{j\ell} = a_{i\ell} + sa_{j\ell}$$

and the $i$th row of $B$ is the same as the $i$th row of $A$. Therefore

$$B = EA$$

and $A \xrightarrow{R_i \rightarrow \mathcal{I} + sR_j} B$, as claimed.
Example 1.4.D

Example 1.4.D. Multiply some $3 \times 3$ matrix $A$ by $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to swap Row 1 and Row 2.

Solution. We have

$$EA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} (0)(1) + (1)(4) + (0)(7) \\ (0)(2) + (1)(5) + (0)(8) \\ (0)(3) + (1)(6) + (1)(9) \\ (1)(1) + (0)(4) + (0)(7) \\ (1)(2) + (0)(5) + (0)(8) \\ (1)(3) + (0)(6) + (0)(9) \\ (0)(1) + (0)(4) + (1)(7) \\ (0)(2) + (0)(5) + (1)(8) \\ (0)(3) + (0)(6) + (1)(9) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}.$$  □

Page 70 Number 44

Page 70 Number 44. Find a matrix $C$ such that

$$C \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}.$$  

Solution. We see that $C$ must be $3 \times 3$, so let $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}.$

Then we need

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}.$$  

or

$$\begin{bmatrix} c_{11} + 3c_{12} + 4c_{13} & 2c_{11} + 4c_{12} + 2c_{13} \\ c_{21} + 3c_{22} + 4c_{23} & 2c_{21} + 4c_{22} + 2c_{23} \\ c_{31} + 3c_{32} + 4c_{33} & 2c_{31} + 4c_{32} + 2c_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}.$$  

Page 70 Number 44 (continued 1)

Solution (continued). We can treat this as three systems of equations based on the rows here:

$$c_{11} + 3c_{12} + 4c_{13} = 1$$
$$c_{21} + 3c_{22} + 4c_{23} = 0$$
$$2c_{31} + 3c_{32} + 4c_{33} = 0$$

$$2c_{11} + 4c_{12} + 2c_{13} = 2$$
$$2c_{21} + 4c_{22} + 2c_{23} = -2$$
$$2c_{31} + 4c_{32} + 2c_{33} = -6.$$  

These three systems of equations have associated augmented matrices:

$$\begin{bmatrix} 1 & 3 & 4 & 1 \\ 2 & 4 & 2 & 2 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{bmatrix} 1 & 3 & 4 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 3R_2} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \end{bmatrix}.$$  

We put each in RREF:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \end{bmatrix} $$

We then need

$$c_{11} - 5c_{13} = 1$$
$$c_{12} + 3c_{13} = 0$$
$$c_{12} = 1 + 5c_{13}$$
$$c_{13} = -3c_{13}.$$  

Page 70 Number 44 (continued 2)

Solution (continued).

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - R_2} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_1} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & -3 \end{bmatrix}.$$  

We put each in RREF:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \end{bmatrix} $$

We then need

$$c_{11} - 5c_{13} = 1$$
$$c_{12} + 3c_{13} = 0$$
$$c_{12} = 1 + 5c_{13}$$
$$c_{13} = -3c_{13}.$$
Solution (continued).

\[
\begin{align*}
-5c_{23} &= -3 & c_{21} &= -3 + 5c_{23} \\
3c_{23} &= 1 & c_{22} &= 1 - 3c_{23} \quad (2) \\
5c_{33} &= -9 & c_{31} &= -9 + 5c_{33} \\
3c_{33} &= 3 & c_{32} &= 3 - 3c_{33} \quad (3) \\
& & c_{33} &= c_{33}
\end{align*}
\]

So in our general solution, \( c_{13} \), \( c_{23} \), and \( c_{33} \) act as free variables. To find matrix \( C \), we can therefore pick any values for \( c_{13} \), \( c_{23} \), and \( c_{33} \) (so there are infinitely many possible choices for \( C \)). The easiest choice is to set \( c_{13} = c_{23} = c_{33} = 0 \) and then we have \( c_{11} = 1, c_{12} = 0, c_{21} = -3 \).

\[
c_{22} = 1, \quad c_{31} = -9, \quad \text{and} \quad c_{32} = 3.
\]

This gives \( C = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -9 & 3 & 0 \end{bmatrix} \).

\[\square\]

Page 70 Number 50

Let \( A \) be a \( 4 \times 4 \) matrix. Find a matrix \( C \) such that the result of applying the sequence of elementary operations:

1. Interchange Row 1 and Row 4,
2. Add 6 times Row 2 to Row 1,
3. Add \(-3\) times Row 1 to Row 3,
4. Add \(-2\) times Row 4 to Row 2,

to \( A \) can also be found by computing the product \( CA \).

Solution. We find the elementary matrices which represent the row operations:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{array}{c}
R_1 \leftrightarrow R_4 \\
R_1 \rightarrow R_1 + 6R_4 \\
R_3 \rightarrow R_3 - 3R_1 \\
R_2 \rightarrow R_2 - 2R_4
\end{array}
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
= E_1,
\]

Then we take \( C = E_4E_3E_2E_1 \):

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 6 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
0 & 6 & 0 & 1 \\
-2 & 1 & 0 & 0 \\
0 & -18 & 1 & -3 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]

\[\square\]
Page 71 Number 56. Find \( a, b, \) and \( c \) such that the parabola \( y = ax^2 + bx + c \) passes through the points \((1, -4), (-1, 0), \) and \((2, 3)\).

Solution. We need \((-4) = a(1)^2 + b(1) + c, \) \((0) = a(-1)^2 + b(-1) + c, \) and \((3) = a(2)^2 + b(2) + c. \) So we have the system of equations
\[
\begin{align*}
a + b + c &= -4 \\
a - b + c &= 0 \\
4a + 2b + c &= 3
\end{align*}
\]
so we consider the augmented matrix and reduce it:
\[
\begin{bmatrix}
1 & 1 & 1 & -4 \\
1 & -1 & 1 & 0 \\
4 & 2 & 1 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 1 & -4 \\
0 & -2 & 0 & 4 \\
0 & -2 & -3 & 19
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 1 & -4 \\
0 & 1 & 0 & -2 \\
0 & -2 & -3 & 19
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 & -2 \\
0 & 1 & 0 & -2 \\
0 & 0 & -3 & 15
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 & -2 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & -5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 3 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & -5
\end{bmatrix}.
\]
So we take \(a = 3, b = -2, c = -5\) and get the parabola \(y = 3x^2 - 2x - 5.\) Note: Just as two distinct points in the plane determine a line, three non-collinear points in a plane determine a parabola. \(\square\)