Chapter 1. Vectors, Matrices, and Linear Systems
Section 1.5. Inverses of Square Matrices—Proofs of Theorems
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Example 1.5.A

Example 1.5.A. It is easy to invert an elementary matrix. For example, suppose $E_1$ interchanges Row 1 and Row 2 of a $3 \times 3$ matrix. Suppose $E_2$ multiplies Row 2 by 7 in a $3 \times 3$ matrix. Find the inverses of $E_1$ and $E_2$.

Solution. We have $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
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Solution. We have $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. To invert the operation of interchanging Row 1 and Row 3 we simply interchange them again. To invert the operation of multiplying Row 2 by 7 we divide Row 2 by 7. So we expect $E_1^{-1} = E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We can easily verify that $E_1E_1^{-1} = \mathcal{I}$ and $E_2E_2^{-1} = \mathcal{I}$. □
Example 1.5.A. It is easy to invert an elementary matrix. For example, suppose $E_1$ interchanges Row 1 and Row 2 of a $3 \times 3$ matrix. Suppose $E_2$ multiplies Row 2 by 7 in a $3 \times 3$ matrix. Find the inverses of $E_1$ and $E_2$.

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Lemma 1.1

Lemma 1.1. Condition for $A\vec{x} = \vec{b}$ to be Solvable for $\vec{b}$.
Let $A$ be an $n \times n$ matrix. The linear system $A\vec{x} = \vec{b}$ has a solution for every choice of column vector $\vec{b} \in \mathbb{R}^n$ if and only if $A$ is row equivalent to the $n \times n$ identity matrix $I$.

Proof. Suppose $A$ is row equivalent to $I$. Let $\vec{b}$ by any column vector in $\mathbb{R}^n$. Then $[A \mid \vec{b}] \sim [I \mid \vec{c}]$ for some column vector $\vec{c} \in \mathbb{R}^n$. Then, by Theorem 1.6, $\vec{x} = \vec{c}$ is a solution to $A\vec{x} = \vec{b}$. 
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Suppose $A$ is not row equivalent to $I$. Row reduce $A$ to a reduced row echelon form $H$ (so $H \neq I$). So the last row (i.e., the $n$th row) of $H$ must be all zeros.
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Lemma 1.1. Condition for $A\vec{x} = \vec{b}$ to be Solvable for $\vec{b}$.

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Proof. Suppose $A$ is row equivalent to $I$. Let $\vec{b}$ by any column vector in $\mathbb{R}^n$. Then $[A | \vec{b}] \sim [I | \vec{c}]$ for some column vector $\vec{c} \in \mathbb{R}^n$. Then, by Theorem 1.6, $\vec{x} = \vec{c}$ is a solution to $A\vec{x} = \vec{b}$.

Suppose $A$ is not row equivalent to $I$. Row reduce $A$ to a reduced row echelon form $H$ (so $H \neq I$). So the last row (i.e., the $n$th row) of $H$ must be all zeros. Now the row reduction of $A$ to $H$ can be accomplished by multiplication on the left by a sequence of elementary matrices by repeated application of Theorem 1.8, “Use of Elementary Matrices.” Say $E_t \cdots E_2 E_1 A = H$. Now elementary matrices are invertible (see Example 1.5.A).
Lemma 1.1

Lemma 1.1. Condition for \( A\vec{x} = \vec{b} \) to be Solvable for \( \vec{b} \).
Let \( A \) be an \( n \times n \) matrix. The linear system \( A\vec{x} = \vec{b} \) has a solution for every choice of column vector \( \vec{b} \in \mathbb{R}^n \) if and only if \( A \) is row equivalent to the \( n \times n \) identity matrix \( \mathcal{I} \).

Proof. Suppose \( A \) is row equivalent to \( \mathcal{I} \). Let \( \vec{b} \) by any column vector in \( \mathbb{R}^n \). Then \([A \mid \vec{b}] \sim [\mathcal{I} \mid \vec{c}]\) for some column vector \( \vec{c} \in \mathbb{R}^n \). Then, by Theorem 1.6, \( \vec{x} = \vec{c} \) is a solution to \( A\vec{x} = \vec{b} \).

Suppose \( A \) is not row equivalent to \( \mathcal{I} \). Row reduce \( A \) to a reduced row echelon form \( H \) (so \( H \neq \mathcal{I} \)). So the last row (i.e., the \( n \)th row) of \( H \) must be all zeros. Now the row reduction of \( A \) to \( H \) can be accomplished by multiplication on the left by a sequence of elementary matrices by repeated application of Theorem 1.8, “Use of Elementary Matrices.” Say \( E_t \cdots E_2 E_1 A = H \). Now elementary matrices are invertible (see Example 1.5.A). Let \( \vec{e}_n \) be the \( n \)th basis element of \( \mathbb{R}^n \) written as a column vector. Define \( \vec{b} = (E_t \cdots E_2 E_1)^{-1}\vec{e}_n \).
Lemma 1.1

Lemma 1.1. Condition for $A\vec{x} = \vec{b}$ to be Solvable for \( \vec{b} \).
Let \( A \) be an \( n \times n \) matrix. The linear system \( A\vec{x} = \vec{b} \) has a solution for every choice of column vector \( \vec{b} \in \mathbb{R}^n \) if and only if \( A \) is row equivalent to the \( n \times n \) identity matrix \( I \).

Proof. Suppose \( A \) is row equivalent to \( I \). Let \( \vec{b} \) by any column vector in \( \mathbb{R}^n \). Then \([A \mid \vec{b}] \sim [I \mid \vec{c}]\) for some column vector \( \vec{c} \in \mathbb{R}^n \). Then, by Theorem 1.6, \( \vec{x} = \vec{c} \) is a solution to \( A\vec{x} = \vec{b} \).

Suppose \( A \) is not row equivalent to \( I \). Row reduce \( A \) to a reduced row echelon form \( H \) (so \( H \neq I \)). So the last row (i.e., the \( n \)th row) of \( H \) must be all zeros. Now the row reduction of \( A \) to \( H \) can be accomplished by multiplication on the left by a sequence of elementary matrices by repeated application of Theorem 1.8, “Use of Elementary Matrices.” Say \( E_t \cdots E_2 E_1 A = H \). Now elementary matrices are invertible (see Example 1.5.A). Let \( \vec{e}_n \) be the \( n \)th basis element of \( \mathbb{R}^n \) written as a column vector. Define \( \vec{b} = (E_t \cdots E_2 E_1)^{-1}\vec{e}_n \).
Lemma 1.1. Condition for $A\vec{x} = \vec{b}$ to be Solvable for $\vec{b}$.

Let $A$ be an $n \times n$ matrix. The linear system $A\vec{x} = \vec{b}$ has a solution for every choice of column vector $\vec{b} \in \mathbb{R}^n$ if and only if $A$ is row equivalent to the $n \times n$ identity matrix $I$.

Proof (continued). Consider the system of equations $A\vec{x} = \vec{b}$ with associated augmented matrix $[A \mid \vec{b}]$. Applying the sequence of elementary row operations associated with $E_t \cdots E_2E_1$ reduces $[A \mid \vec{b}]$ to

$$[E_t \cdots E_2E_1 A \mid E_t \cdots E_2E_1 \vec{b}] = [E_t \cdots E_2E_1 A \mid (E_t \cdots E_2E_1)(E_t \cdots E_2E_1)^{-1} \vec{e}_n]$$

$$= [H \mid \vec{e}_n].$$

But then the last row of $H$ consists of all zeros to the left of the partition and 1 to the right of the partition. So by Theorem 1.7(1), “Solutions of $A\vec{x} = \vec{b}$,” $A\vec{x} = \vec{b}$ has no solution. So if $A$ is not row equivalent to $I$ then the system $A\vec{x} = \vec{b}$ does not have a solution for all $\vec{b} \in \mathbb{R}^n$. $\square$
Lemma 1.1 (continued)

Lemma 1.1. Condition for $A\vec{x} = \vec{b}$ to be Solvable for $\vec{b}$.
Let $A$ be an $n \times n$ matrix. The linear system $A\vec{x} = \vec{b}$ has a solution for every choice of column vector $\vec{b} \in \mathbb{R}^n$ if and only if $A$ is row equivalent to the $n \times n$ identity matrix $\mathcal{I}$.

Proof (continued). Consider the system of equations $A\vec{x} = \vec{b}$ with associated augmented matrix $[A \mid \vec{b}]$. Applying the sequence of elementary row operations associated with $E_t \cdot \cdot \cdot E_2 E_1$ reduces $[A \mid \vec{b}]$ to

$$[E_t \cdot \cdot \cdot E_2 E_1 A \mid E_t \cdot \cdot \cdot E_2 E_1 \vec{b}] = [E_t \cdot \cdot \cdot E_2 E_1 A \mid (E_t \cdot \cdot \cdot E_2 E_1)(E_t \cdot \cdot \cdot E_2 E_1)^{-1} \vec{e}_n]$$

$$= [H \mid \vec{e}_n].$$

But then the last row of $H$ consists of all zeros to the left of the partition and 1 to the right of the partition. So by Theorem 1.7(1), “Solutions of $A\vec{x} = \vec{b}$,” $A\vec{x} = \vec{b}$ has no solution. So if $A$ is not row equivalent to $\mathcal{I}$ then the system $A\vec{x} = \vec{b}$ does not have a solution for all $\vec{b} \in \mathbb{R}^n$. □
Determine whether the span of the column vectors of
\[
A = \begin{bmatrix}
1 & -2 & 1 & 0 \\
-3 & 5 & 0 & 2 \\
0 & 1 & 2 & -4 \\
-1 & 2 & 4 & -2
\end{bmatrix}
\]
span \( \mathbb{R}^4 \).

**Solution.** Recall that for any \( \vec{x} \in \mathbb{R}^n \), \( A\vec{x} \) is a linear combination of the columns of \( A \) by Note 1.3.A. So to see if the column vectors of \( A \) span \( \mathbb{R}^4 \), we need to choose an arbitrary \( \vec{b} \in \mathbb{R}^4 \) and see if there is \( \vec{x} \in \mathbb{R}^4 \) such that \( A\vec{x} = \vec{b} \). That is, we need to see if \( A\vec{x} = \vec{b} \) has a solution for every \( \vec{b} \in \mathbb{R}^4 \).
Determine whether the span of the column vectors of \( A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -3 & 5 & 0 & 2 \\ 0 & 1 & 2 & -4 \\ -1 & 2 & 4 & -2 \end{bmatrix} \) span \( \mathbb{R}^4 \).

**Solution.** Recall that for any \( \vec{x} \in \mathbb{R}^n \), \( A\vec{x} \) is a linear combination of the columns of \( A \) by Note 1.3.A. So to see if the column vectors of \( A \) span \( \mathbb{R}^4 \), we need to choose an arbitrary \( \vec{b} \in \mathbb{R}^4 \) and see if there is \( \vec{x} \in \mathbb{R}^4 \) such that \( A\vec{x} = \vec{b} \). That is, we need to see if \( A\vec{x} = \vec{b} \) has a solution for every \( \vec{b} \in \mathbb{R}^4 \).

So by Lemma 1.1 we only need to see if \( A \) is row equivalent to \( I \). Consider

\[
A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -3 & 5 & 0 & 2 \\ 0 & 1 & 2 & -4 \\ -1 & 2 & 4 & -2 \end{bmatrix}
\]

\[
\begin{array}{c}
R_2 \rightarrow R_2 + 3R_1 \\
R_4 \rightarrow R_4 + R_1
\end{array}
\]

\[
\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & -1 & 3 & 2 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 5 & -2 \end{bmatrix}
\]
**Page 84 Number 12**

**Page 84 Number 12.** Determine whether the span of the column vectors of 
\[
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1 & -2 & 1 & 0 \\
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\] 
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**Solution.** Recall that for any \( \vec{x} \in \mathbb{R}^n \), \( A\vec{x} \) is a linear combination of the columns of \( A \) by Note 1.3.A. So to see if the column vectors of \( A \) span \( \mathbb{R}^4 \), we need to choose an arbitrary \( \vec{b} \in \mathbb{R}^4 \) and see if there is \( \vec{x} \in \mathbb{R}^4 \) such that \( A\vec{x} = \vec{b} \). That is, we need to see if \( A\vec{x} = \vec{b} \) has a solution for every \( \vec{b} \in \mathbb{R}^4 \). So by Lemma 1.1 we only need to see if \( A \) is row equivalent to \( I \). Consider

\[
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1 & -2 & 1 & 0 \\
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-1 & 2 & 4 & -2 \\
\end{bmatrix}
\]

\[
\begin{array}{c}
R_2 \rightarrow R_2 + 3R_1 \\
R_4 \rightarrow R_4 + R_1 \\
\end{array}
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & -1 & 3 & 2 \\
0 & 1 & 2 & -4 \\
0 & 0 & 5 & -2 \\
\end{bmatrix}
\]
Solution (continued).

\[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & -1 & 3 & 2 \\
0 & 1 & 2 & -4 \\
0 & 0 & 5 & -2
\end{bmatrix}
\]

\[
\begin{array}{c}
R_1 \rightarrow R_1 - 2R_2 \\
R_3 \rightarrow R_3 + R_2
\end{array}
\]

\[
\begin{bmatrix}
1 & 0 & -5 & -4 \\
0 & -1 & 3 & 2 \\
0 & 0 & 5 & -2
\end{bmatrix}
\]

Now \( H \) is in reduced row echelon form and \( H \neq I \). So Lemma 1.1 implies that NO, the columns do not span \( \mathbb{R}^4 \).

\[\square\]
Solution (continued).

Now $H$ is in reduced row echelon form and $H \neq I$. So Lemma 1.1 implies that NO, the columns do not span $\mathbb{R}^4$. □
Solution (continued).

\[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & -1 & 3 & 2 \\
0 & 1 & 2 & -4 \\
0 & 0 & 5 & -2
\end{bmatrix}
\]

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R_3 \rightarrow R_3 + R_2
\end{array}
\]

\[
\begin{bmatrix}
1 & 0 & -5 & -4 \\
0 & -1 & 3 & 2 \\
0 & 0 & 5 & -2
\end{bmatrix}
\]

\[
\begin{array}{c}
R_1 \rightarrow R_1 + R_3 \\
R_2 \rightarrow R_2 - (3/5)R_3 \\
R_4 \rightarrow R_4 - R_3
\end{array}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & -6 \\
0 & -1 & 0 & 16/5 \\
0 & 0 & 5 & -2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{array}{c}
R_2 \rightarrow R_2 \\
R_3 \rightarrow R_3/5
\end{array}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & -6 \\
0 & 1 & 0 & -16/5 \\
0 & 0 & 1 & -2/5 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[= H.\]

Now \(H\) is in reduced row echelon form and \(H \neq I\). So Lemma 1.1 implies that \[\text{NO, the columns do not span } \mathbb{R}^4. \]
Theorem 1.11. A Commutivity Property.

Let $A$ and $C$ be $n \times n$ matrices. Then $CA = I$ if and only if $AC = I$.

**Proof.** Suppose that $AC = I$. Then the equation $A\vec{x} = \vec{b}$ has a solution for every column vector $\vec{b} \in \mathbb{R}^n$. Notice that $\vec{x} = C\vec{b}$ is a solution because

$$A(C\vec{b}) = (AC)\vec{b} = I\vec{b} = \vec{b}.$$
Theorem 1.11. A Commutivity Property.
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$$A(C\vec{b}) = (AC)\vec{b} = I\vec{b} = \vec{b}.$$ 

By Lemma 1.1, we know that $A$ is row equivalent to the $n \times n$ identity matrix $I$, and so there exists a sequence of elementary matrices $E_1, E_2, \ldots, E_t$ such that $(E_t \cdots E_2 E_1)A = I$. By Theorem 1.9, the two equations

$$(E_t \cdots E_2 E_1)A = I \text{ and } AC = I$$

imply that $E_t \cdots E_2 E_1 = C$, and so we have $CA = I$. 
Theorem 1.11. A Commutivity Property.
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$$A(C\vec{b}) = (AC)\vec{b} = I\vec{b} = \vec{b}.$$ 

By Lemma 1.1, we know that $A$ is row equivalent to the $n \times n$ identity matrix $I$, and so there exists a sequence of elementary matrices $E_1, E_2, \ldots, E_t$ such that $(E_t \cdots E_2 E_1)A = I$. By Theorem 1.9, the two equations

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$$A(C\vec{b}) = (AC)\vec{b} = I\vec{b} = \vec{b}.$$ 

By Lemma 1.1, we know that $A$ is row equivalent to the $n \times n$ identity matrix $I$, and so there exists a sequence of elementary matrices $E_1, E_2, \ldots, E_t$ such that $(E_t \cdots E_2 E_1)A = I$. By Theorem 1.9, the two equations

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imply that $E_t \cdots E_2 E_1 = C$, and so we have $CA = I$. The other half of the proof follows by interchanging the roles of $A$ and $C$. $\square$
Consider $A = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$. Find $A^{-1}$. Use $A^{-1}$ to solve the system

\[
\begin{align*}
6x_1 + 7x_2 &= 4 \\
8x_1 + 9x_2 &= 6.
\end{align*}
\]

Solution. We form $[A|I]$ and apply Gauss-Jordan elimination to produce the row equivalent $[I|A^{-1}]$ (if possible).
Page 84 Number 4. Consider $A = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$. Find $A^{-1}$. Use $A^{-1}$ to solve the system

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8x_1 + 9x_2 &= 6.
\end{align*}

Solution. We form $[A|I]$ and apply Gauss-Jordan elimination to produce the row equivalent $[I|A^{-1}]$ (if possible). So

\[
\begin{bmatrix}
6 & 7 & 1 & 0 \\
8 & 9 & 0 & 1
\end{bmatrix} \xrightarrow{R_1 \rightarrow R_1/6} \begin{bmatrix}
1 & 7/6 & 1/6 & 0 \\
8 & 9 & 0 & 1
\end{bmatrix}
\]
Page 84 Number 4

**Page 84 Number 4.** Consider \( A = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} \). Find \( A^{-1} \). Use \( A^{-1} \) to solve the system

\[
\begin{align*}
6x_1 + 7x_2 &= 4 \\
8x_1 + 9x_2 &= 6
\end{align*}
\]

**Solution.** We form \([A|I]\) and apply Gauss-Jordan elimination to produce the row equivalent \([I|A^{-1}]\) (if possible). So

\[
\begin{bmatrix}
6 & 7 & | & 1 & 0 \\
8 & 9 & | & 0 & 1
\end{bmatrix}
\xrightarrow{R_1 \rightarrow \frac{1}{6} R_1}
\begin{bmatrix}
1 & \frac{7}{6} & | & \frac{1}{6} & 0 \\
8 & 9 & | & 0 & 1
\end{bmatrix}
\xrightarrow{R_2 \rightarrow R_2 - 8R_1}
\begin{bmatrix}
1 & \frac{7}{6} & | & \frac{1}{6} & 0 \\
0 & -\frac{2}{3} & | & 0 & \frac{1}{3}
\end{bmatrix}
\xrightarrow{-3R_2}
\begin{bmatrix}
1 & \frac{7}{6} & | & \frac{1}{6} & 0 \\
0 & 1 & | & 0 & 1
\end{bmatrix}
\]

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Page 84 Number 4. Consider $A = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$. Find $A^{-1}$. Use $A^{-1}$ to solve the system

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8x_1 + 9x_2 &= 6.
\end{align*}
\]

**Solution.** We form $[A|I]$ and apply Gauss-Jordan elimination to produce the row equivalent $[I|A^{-1}]$ (if possible). So

\[
\begin{bmatrix}
6 & 7 & 1 & 0 \\
8 & 9 & 0 & 1
\end{bmatrix} \xrightarrow{R_1 \rightarrow R_1/6} \begin{bmatrix}
1 & 7/6 & 1/6 & 0 \\
8 & 9 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 7/6 & 1/6 & 0 \\
8 - 8(1) & 9 - 8(7/6) & 0 - 8(1/6) & 1 - 8(0)
\end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - 8R_1} \begin{bmatrix}
1 & 7/6 & 1/6 & 0 \\
0 & -1/3 & -4/3 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 7/6 & 1/6 & 0 \\
0 & 1 & 4 & -3
\end{bmatrix} \xrightarrow{R_2 \rightarrow -3R_2}
\]
Consider $A = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$. Find $A^{-1}$. Use $A^{-1}$ to solve the system

\begin{align*}
6x_1 + 7x_2 &= 4 \\
8x_1 + 9x_2 &= 6.
\end{align*}

**Solution.** We form $[A|I]$ and apply Gauss-Jordan elimination to produce the row equivalent $[I|A^{-1}]$ (if possible). So

\[
\begin{bmatrix}
6 & 7 & 1 & 0 \\
8 & 9 & 0 & 1
\end{bmatrix}
\overset{R_1 \rightarrow R_1/6}{\sim}
\begin{bmatrix}
1 & 7/6 & 1/6 & 0 \\
8 & 9 & 0 & 1
\end{bmatrix}
\]

\[
R_2 \rightarrow R_2 - 8R_1
\begin{bmatrix}
1 & 7/6 & 1/6 & 0 \\
8 - 8(1) & 9 - 8(7/6) & 0 - 8(1/6) & 1 - 8(0)
\end{bmatrix}
\]

\[= \begin{bmatrix} 1 & 7/6 & 1/6 & 0 \\ 0 & -1/3 & -4/3 & 1 \end{bmatrix} \overset{R_2 \rightarrow -3R_2}{\sim} \begin{bmatrix} 0 & 1 & 4 & -3 \end{bmatrix}
\]
Solution (continued).

\[
\begin{bmatrix}
1 & 7/6 & 1/6 & 0 \\
0 & 1 & 4 & -3 \\
\end{bmatrix}
\xrightarrow{R_1 \rightarrow R_1 - (7/6)R_2}
\begin{bmatrix}
1 - (7/6)(0) & 7/6 - (7/6)(1) & 1/6 - (7/6)(4) & 0 - (7/6)(-3) \\
0 & 1 & 4 & -3 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & -9/2 & 7/2 \\
0 & 1 & 4 & -3 \\
\end{bmatrix}.
\]

So
\[
A^{-1} = \begin{bmatrix}
-9/2 & 7/2 \\
4 & -3 \\
\end{bmatrix}.
\]
Solution (continued).

\[
\begin{bmatrix}
1 & \frac{7}{6} & \frac{1}{6} & 0 \\
0 & 1 & 4 & -3
\end{bmatrix}
\overset{R_1 \rightarrow R_1 - \left(\frac{7}{6}\right) R_2}{\longrightarrow}
\begin{bmatrix}
1 - \left(\frac{7}{6}\right)(0) & \frac{7}{6} - \left(\frac{7}{6}\right)(1) & \frac{1}{6} - \left(\frac{7}{6}\right)(4) & 0 - \left(\frac{7}{6}\right)(-3) \\
0 & 1 & 4 & -3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & -\frac{9}{2} & \frac{7}{2} \\
0 & 1 & 4 & -3
\end{bmatrix}.
\]

So \( A^{-1} = \begin{bmatrix} -\frac{9}{2} & \frac{7}{2} \\ 4 & -3 \end{bmatrix}. \)
Solution (continued). For the system of equations, we express it as a matrix product $A\vec{x} = \vec{b}$: 

$$
\begin{bmatrix}
6 & 7 \\
8 & 9
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
4 \\
6
\end{bmatrix}.
$$

Then $A^{-1}A\vec{x} = A^{-1}\vec{b}$ or $I\vec{x} = A^{-1}\vec{b}$ or $\vec{x} = A^{-1}\vec{b}$. So

$$
\vec{x} = 
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = A^{-1}\vec{b} =
\begin{bmatrix}
-9/2 & 7/2 \\
4 & -3
\end{bmatrix}
\begin{bmatrix}
4 \\
6
\end{bmatrix}
$$

$$
= 
\begin{bmatrix}
(-9/2)(4) + (7/2)(6) \\
4(4) - 3(6)
\end{bmatrix} =
\begin{bmatrix}
3 \\
-2
\end{bmatrix}
$$

and the solution is $x_1 = 3$, $x_2 = -2$. □
Solution (continued). For the system of equations, we express it as a matrix product $A\vec{x} = \vec{b}$: 

$$
\begin{bmatrix}
6 & 7 \\
8 & 9 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
6 \\
\end{bmatrix}.
$$

Then $A^{-1}A\vec{x} = A^{-1}\vec{b}$ or $I\vec{x} = A^{-1}\vec{b}$ or $\vec{x} = A^{-1}\vec{b}$. So

$$
\vec{x} = 
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix} = A^{-1}\vec{b} = 
\begin{bmatrix}
-9/2 & 7/2 \\
4 & -3 \\
\end{bmatrix}
\begin{bmatrix}
4 \\
6 \\
\end{bmatrix}
$$

$$
= 
\begin{bmatrix}
(-9/2)(4) + (7/2)(6) \\
4(4) - 3(6) \\
\end{bmatrix}
= 
\begin{bmatrix}
3 \\
-2 \\
\end{bmatrix}
$$

and the solution is $x_1 = 3, x_2 = -2$. □
Page 85 number 24. Prove that if $A$ is an invertible $n \times n$ matrix then $A^T$ is invertible. Describe $(A^T)^{-1}$ in terms of $A^{-1}$.

Solution. We know that $(AB)^T = B^T A^T$ (see “Properties of the Transpose Operator” in Section 1.3; page 4 of the notes).
**Page 85 number 24.** Prove that if $A$ is an invertible $n \times n$ matrix then $A^T$ is invertible. Describe $(A^T)^{-1}$ in terms of $A^{-1}$.

**Solution.** We know that $(AB)^T = B^T A^T$ (see “Properties of the Transpose Operator” in Section 1.3; page 4 of the notes). Since $A$ is invertible then $AA^{-1} = A^{-1} A = \mathcal{I}$. So $(AA^{-1})^T = (A^{-1} A)^T = \mathcal{I}^T = \mathcal{I}$ (since the identity matrix $\mathcal{I}$ is symmetric; see Definition 1.11).
Page 85 number 24. Prove that if $A$ is an invertible $n \times n$ matrix then $A^T$ is invertible. Describe $(A^T)^{-1}$ in terms of $A^{-1}$.

Solution. We know that $(AB)^T = B^T A^T$ (see “Properties of the Transpose Operator” in Section 1.3; page 4 of the notes). Since $A$ is invertible then $AA^{-1} = A^{-1} A = I$. So $(AA^{-1})^T = (A^{-1} A)^T = I^T = I$ (since the identity matrix $I$ is symmetric; see Definition 1.11). Hence $(A^{-1})^T A^T = A^T (A^{-1})^T = I$ and so the inverse of $A^T$ is $(A^{-1})^T$. Therefore $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$. □
Page 85 number 24. Prove that if $A$ is an invertible $n \times n$ matrix then $A^T$ is invertible. Describe $(A^T)^{-1}$ in terms of $A^{-1}$.

Solution. We know that $(AB)^T = B^T A^T$ (see “Properties of the Transpose Operator” in Section 1.3; page 4 of the notes). Since $A$ is invertible then $AA^{-1} = A^{-1} A = I$. So $(AA^{-1})^T = (A^{-1} A)^T = I^T = I$ (since the identity matrix $I$ is symmetric; see Definition 1.11). Hence $(A^{-1})^T A^T = A^T (A^{-1})^T = I$ and so the inverse of $A^T$ is $(A^{-1})^T$. Therefore $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$. □
A square matrix $A$ is said to be idempotent if $A^2 = A$.

(a) Give an example of an idempotent matrix other than 0 and $I$.

**Solution.** An easy example can be found by slightly modifying $I$.

Consider, say, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A.$$
A square matrix $A$ is said to be *idempotent* if $A^2 = A$.

**Solution.** An easy example can be found by slightly modifying $I$.

Consider, say, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A.$$

**Proof.** Suppose $A^2 = A$ and $A^{-1}$ exists. Then $A^{-1}(A^2) = A^{-1}A$ and by associativity (Theorem 1.3.A(8)), $(A^{-1}A)A = A^{-1}A$ or $I_A = I$ or $A = I$.

(b) Prove that if matrix $A$ is both idempotent and invertible, then $A = I$. 

A square matrix $A$ is said to be *idempotent* if $A^2 = A$.

**(a)** Give an example of an idempotent matrix other than 0 and $I$.

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$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A.$$ 

**(b)** Prove that if matrix $A$ is both idempotent and invertible, then $A = I$.

**Proof.** Suppose $A^2 = A$ and $A^{-1}$ exists. Then $A^{-1}(A^2) = A^{-1}A$ and by associativity (Theorem 1.3.A(8)), "Properties of Matrix Algebra") $(A^{-1}A)A = A^{-1}A$ or $IA = I$ or $A = I$. 

\[ \square \]
A square matrix \( A \) is said to be idempotent if \( A^2 = A \).

(a) Give an example of an idempotent matrix other than 0 and \( I \).

**Solution.** An easy example can be found by slightly modifying \( I \).

Consider, say, \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \). Then

\[
A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A.
\]

(b) Prove that if matrix \( A \) is both idempotent and invertible, then \( A = I \).

**Proof.** Suppose \( A^2 = A \) and \( A^{-1} \) exists. Then \( A^{-1}(A^2) = A^{-1}A \) and by associativity (Theorem 1.3.A(8)), “Properties of Matrix Algebra”) \( (A^{-1}A)A = A^{-1}A \) or \( IA = I \) or \( A = I \).