Chapter 1. Vectors, Matrices, and Linear Systems
Section 1.6. Homogeneous Systems, Subspaces, and Bases—Proofs of Theorems

Page 99 Number 8. Determine whether the set
\[ W = \{ [2x, x + y, y] \mid x, y \in \mathbb{R} \} \]
is a subspace of \( \mathbb{R}^3 \).

**Solution.** By Definition 1.16, we need to see if \( W \) is closed under vector addition and scalar multiplication. Let \( \vec{v}_1, \vec{v}_2 \in W \). Then \( \vec{v}_1 = [2x_1, x_1 + y_1, y_1] \) and \( \vec{v}_2 = [2x_2, x_2 + y_2, y_2] \) for some \( x_1, x_2, y_1, y_2 \in \mathbb{R} \). For vector addition,
\[
\vec{v}_1 + \vec{v}_2 = [2x_1, x_1 + y_1, y_1] + [2x_2, x_2 + y_2, y_2] \\
= [2x_1 + 2x_2, (x_1 + y_1) + (x_2 + y_2), (y_1 + y_2)] \\
= [2(x_1 + x_2), (x_1 + y_2) + (y_1 + y_2)] \\
= [2x, x + y, y] \text{ where } x = x_1 + x_2 \text{ and } y = y_1 + y_2.
\]
So \( \vec{v}_1 + \vec{v}_2 \in W \) and \( W \) is closed under vector addition.

**Page 99 Number 8 (continued)**

**Theorem 1.14.** Subspace Property of a Span

Let \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) be the span of \( k > 0 \) vectors in \( \mathbb{R}^n \). Then \( W \) is a subspace of \( \mathbb{R}^n \). (The vectors \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \) are said to span or generate the subspace.)

**Proof.** We use Definition 1.16, “Closure and Subspace.” Let \( \vec{u}, \vec{v} \in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) and let \( c \) be a scalar. Then
\[
\vec{u} = r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k \quad \text{and} \quad \vec{v} = s_1 \vec{w}_1 + s_2 \vec{w}_2 + \cdots + s_k \vec{w}_k
\]
for some scalars \( r_i \) and \( s_i \). Then
\[
\vec{u} + \vec{v} = (r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k) + (s_1 \vec{w}_1 + s_2 \vec{w}_2 + \cdots + s_k \vec{w}_k)
\]
\[
= (r_1 + s_1) \vec{w}_1 + (r_2 + s_2) \vec{w}_2 + \cdots + (r_k + s_k) \vec{w}_k \text{ by } S1 \text{ and } S2
\]
\[
\in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)
\]
and so \( \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) is closed under vector addition.
Theorem 1.14. Subspace Property of a Span
Let \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) be the span of \( k > 0 \) vectors in \( \mathbb{R}^n \). Then \( W \) is a subspace of \( \mathbb{R}^n \). (The vectors \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \) are said to span or generate the subspace.)

Proof (continued). Next,
\[
    c\vec{u} = c(r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k) \\
    = (cr_1)\vec{w}_1 + (cr_2)\vec{w}_2 + \cdots + (cr_k)\vec{w}_k \text{ by S1 and S3} \\
    \in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)
\]
and so \( \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) is closed under scalar multiplication. So by Definition 1.16 \( \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) is a subspace of \( \mathbb{R}^n \). \( \square \)

Page 100 Number 18

Page 100 Number 18 (continued 1)

Solution (continued). \[
\begin{bmatrix}
1 & 0 & 2 & -1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & -4/5 & 0
\end{bmatrix}
\]
\( R_1 \rightarrow R_1 - 2R_3 \\
R_2 \rightarrow R_2 - R_3 \\
R_3 \rightarrow -R_3 - 3R_2 \)

Returning to a system of equations,
\[
\begin{align*}
x_1 + (3/5)x_4 &= 0 \\
x_2 + (4/5)x_4 &= 0 \\
x_3 - (4/5)x_4 &= 0 \\
x_4 &= x_4.
\end{align*}
\]

So let \( r = x_4 \) be a free variable and we have that the general solution is of the form
\[
\vec{x} \in \left\{ \begin{bmatrix} -3/5 \\ -4/5 \\ 4/5 \\ 1 \\ r \end{bmatrix} \middle| r \in \mathbb{R} \right\}.
\]

Page 100 Number 18 (continued 2)

Solution (continued). So a generating set for the system is
\[
\left\{ \begin{bmatrix} -3/5 \\ -4/5 \\ 4/5 \\ 1 \end{bmatrix} \right\}.
\]

Note: We could have let \( s = x_4/5 \) be a free variable in which case a generating set is given by the simpler \( \left\{ \begin{bmatrix} -3 \\ -4 \\ 4 \\ 5 \end{bmatrix} \right\} \). \( \square \)
Theorem 1.15. Unique Linear Combinations.
The set \( \{ \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_k \} \) is a basis for \( W = \text{sp}(\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_k) \) if and only if
\[
r_1 \tilde{w}_1 + r_2 \tilde{w}_2 + \cdots + r_k \tilde{w}_k = \tilde{0}
\]
implies
\[
r_1 = r_2 = \cdots = r_k = 0.
\]

Proof. Suppose \( \{ \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_k \} \) is a basis for \( W = \text{sp}(\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_k) \).
Then by Definition 1.17, “Basis for a Subspace,” every vector in \( W \) can be expressed uniquely as a linear combination of the \( \tilde{w}_i \). In particular,
\[
r_1 \tilde{w}_1 + r_2 \tilde{w}_2 + \cdots + r_k \tilde{w}_k = \tilde{0}
\]
is satisfied for \( r_1 = r_2 = \cdots = r_k = 0 \) and the uniqueness condition implies that we must have \( r_1 = r_2 = \cdots = r_k = 0 \).

Page 100 Number 22(a)

Page 100 Number 22(a). Use Theorem 1.15 to determine whether the set \( \{[-1,1],[1,2]\} \) is a basis for the subspace of \( \mathbb{R}^2 \) that it spans.

Solution. Based on Theorem 1.15, we consider scalars \( r_1, r_2 \in \mathbb{R} \) such that \( r_1[-1,1] + r_2[1,2] = [0,0] \). This implies \([-r_1, r_1] + [r_2, 2r_2] = [0,0] \) or \([-r_1 + r_2, r_1 + 2r_2] = [0,0] \). So we need
\[
-r_1 + r_2 = 0 \quad (1) \\
-r_1 + 2r_2 = 0. \quad (2)
\]
From (1) we see that \( r_1 = r_2 \) and so from (2) we need \( r_1 + 2(r_1) = 0 \) or \( 3r_1 = 0 \) or \( r_1 = 0 \). Since \( r_1 = r_2 \) we also need \( r_2 = 0 \). Hence we must have \( r_1 = r_2 = 0 \) and so \( \{[-1,1],[1,2]\} \) is a basis for its span by Theorem 1.15.

Page 100 Number 22(b)

Page 100 Number 22(b). Use Theorem 1.16 to determine whether the set \( \{[-1,1],[1,2]\} \) is a basis for the subspace of \( \mathbb{R}^2 \) that it spans.

Solution. We define matrix \( A \) which has as its columns the vectors in the set: \( A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \). By Theorem 1.16, we see that the columns of \( A \) form a basis for \( \mathbb{R}^2 \) if and only if \( A \) is row equivalent to \( I \). So we row reduce \( A \):
\[
A \sim \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} R_2 \rightarrow -R_2 \\ R_2 \rightarrow R_2 + R_1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} R_2 \rightarrow R_2 \\ R_2 \rightarrow R_2 + R_1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = I.
\]
So \( A \sim I \) and hence the columns of \( A \) form a basis for \( \mathbb{R}^2 \); that is, the set \( \{[-1,1],[1,2]\} \) is a basis for the subspace of \( \mathbb{R}^2 \) that it spans. (Since there are two vectors, their span is all of \( \mathbb{R}^2 \).)
**Example.** Page 97 Example 6. A basis of \( \mathbb{R}^n \) cannot contain more than \( n \) vectors.

**Proof.** Suppose \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \) is a basis for \( \mathbb{R}^n \) and ASSUME \( k > n \). Consider the system \( A\vec{x} = \vec{0} \) where the column vectors of \( A \) are \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \). Then \( A \) has \( n \) rows and \( k \) columns (corresponding to \( n \) equations in \( k \) unknowns). With \( n < k \), Corollary 2 implies there is a nontrivial solution to \( A\vec{x} = \vec{0} \). But this corresponds to a linear combination of the columns of \( A \) which equals \( \vec{0} \) while not all the coefficients are 0. This CONTRADS Theorem 1.15 (since we then have two different linear combinations of \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \) which equal \( \vec{0} \)). So the assumption that \( k > n \) is false. Therefore, \( k \leq n \). \( \square \)

**Theorem 1.18.** Structure of the Solution Set of \( A\vec{x} = \vec{b} \).

Let \( A\vec{x} = \vec{b} \) be a linear system. If \( \vec{p} \) is any particular solution of \( A\vec{x} = \vec{b} \) and \( \vec{h} \) is a solution to \( A\vec{x} = \vec{0} \), then \( \vec{p} + \vec{h} \) is a solution of \( A\vec{x} = \vec{b} \). In fact, every solution of \( A\vec{x} = \vec{b} \) has the form \( \vec{p} + \vec{h} \) and the general solution is \( \vec{x} = \vec{p} + \vec{h} \) where \( A\vec{h} = \vec{0} \) (that is, \( \vec{h} \) is an arbitrary element of the nullspace of \( A \)).

**Proof.** Let \( \vec{p} \) be a particular solution of \( A\vec{x} = \vec{b} \) (so that \( A\vec{p} = \vec{b} \)). Let \( \vec{h} \) be a solution of the homogeneous system \( A\vec{x} = \vec{0} \) (so that \( A\vec{h} = \vec{0} \)). Then

\[
A(\vec{p} + \vec{h}) = A\vec{p} + A\vec{h} \text{ by Theorem 1.2.A(10),}
\]

**Distribution of Matrix Multiplication**

\[
= \vec{b} + \vec{0} = \vec{b}.
\]

So \( \vec{p} + \vec{h} \) is a solution of \( A\vec{x} = \vec{b} \).

**Theorem 1.18 (continued)**

**Theorem 1.18.** Structure of the Solution Set of \( A\vec{x} = \vec{b} \).

Let \( A\vec{x} = \vec{b} \) be a linear system. If \( \vec{p} \) is any particular solution of \( A\vec{x} = \vec{b} \) and \( \vec{h} \) is a solution to \( A\vec{x} = \vec{0} \), then \( \vec{p} + \vec{h} \) is a solution of \( A\vec{x} = \vec{b} \). In fact, every solution of \( A\vec{x} = \vec{b} \) has the form \( \vec{p} + \vec{h} \) and the general solution is \( \vec{x} = \vec{p} + \vec{h} \) where \( A\vec{h} = \vec{0} \) (that is, \( \vec{h} \) is an arbitrary element of the nullspace of \( A \)).

**Proof (continued).** Now suppose \( \vec{q} \) is any solution to \( A\vec{x} = \vec{b} \). With \( \vec{p} \) as a particular solution to \( A\vec{x} = \vec{b} \) we have

\[
A(\vec{q} - \vec{p}) = A\vec{q} - A\vec{p} \text{ by Theorem 1.2.A(10),}
\]

**Distribution of Matrix Multiplication**

\[
= \vec{b} - \vec{b} = \vec{0}.
\]

So \( \vec{q} - \vec{p} \) is a solution of \( A\vec{x} = \vec{0} \), say \( \vec{q} - \vec{p} = \vec{h} \). So \( \vec{q} = \vec{p} + \vec{h} \) and every solution \( \vec{x} \) of \( A\vec{x} = \vec{b} \) is of the form \( \vec{p} + \vec{h} \) where \( \vec{p} \) is a particular solution of \( A\vec{x} = \vec{b} \) and \( \vec{h} \) is any solution of \( A\vec{x} = \vec{0} \). \( \square \)

**Page 100 Number 36.** Solve the linear system

\[
\begin{align*}
x_1 & - 2x_2 + x_3 + x_4 = 4 \\
2x_1 & + x_2 - 3x_3 - x_4 = 6 \\
x_1 & - 7x_2 - 6x_3 + 2x_4 = 6
\end{align*}
\]

and express the solution set in a form that illustrates Theorem 1.18.

**Solution.** We apply Gauss-Jordan elimination to the augmented matrix:

\[
\begin{bmatrix}
1 & -2 & 1 & 1 & 4 \\
2 & 1 & -3 & -1 & 6 \\
1 & -7 & 6 & 2 & 6
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -2 & 1 & 1 & 4 \\
0 & 5 & -5 & -3 & -2 \\
0 & 0 & 1 & 1/6 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & -2/5 & 4 \\
0 & 1 & -3/5 & -2/5 & 0 \\
0 & 0 & 1 & 1/6 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 4 & 0 \\
0 & 1 & 0 & -2/5 & 0 \\
0 & 0 & 1 & 1/6 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 4 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

Thus, the solution set is given by

\[
x_1 = 4, \quad x_2 = 0, \quad x_3 = 0.
\]

Theorem 1.18 (continued)
Solution (continued).

\[
R_1 \rightarrow R_1 + 2R_2
\begin{bmatrix}
1 & 0 & -1 & -1/5 & 16/5 \\
0 & 1 & -1 & -3/5 & -2/5 \\
0 & 0 & 1 & 1/6 & 0
\end{bmatrix}
\]

\[
R_3 \rightarrow R_2 + R_3
\begin{bmatrix}
1 & 0 & 0 & -1/30 & 16/5 \\
0 & 1 & 0 & -13/30 & -2/5 \\
0 & 0 & 1 & 1/6 & 0
\end{bmatrix}
\]

This corresponds to the system of equations:

\[
\begin{align*}
x_1 & = -(1/30)x_4 = 16/5 \\
x_2 & = -(13/30)x_4 = -2/5 \\
x_3 & = (1/6)x_4 = 0
\end{align*}
\]

For a particular solution \( \vec{p} \) to the original system of equations we choose to set \( x_4 = 0 \) so that...
Page 101 Number 47. Let $W_1$ and $W_2$ be two subspaces of $\mathbb{R}^n$. Prove that their intersection $W_1 \cap W_2$ is also a subspace of $\mathbb{R}^n$.

**Proof.** We use Definition 1.16, “Closure and Subspace.” Let $\vec{u}, \vec{v} \in W_1 \cap W_2$. Since $W_1$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_1$. Since $W_2$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_2$. Hence $\vec{u} + \vec{v}$ is in both $W_1$ and $W_2$; that is, $\vec{u} + \vec{v} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under vector addition.

Now let $r$ be a scalar. Since $W_1$ is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_1$. Since $W_2$ is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_2$. Hence $r\vec{u}$ is in both $W_1$ and $W_2$; that is, $r\vec{u} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under scalar multiplication. By Definition 1.16, $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$. \qed