Chapter 1. Vectors, Matrices, and Linear Systems
Section 1.6. Homogeneous Systems, Subspaces, and Bases—Proofs of Theorems
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Problem 99 Number 8. Determine whether the set
\[ W = \{ [2x, x + y, y] \mid x, y \in \mathbb{R} \} \] is a subspace of \( \mathbb{R}^3 \).

Solution. By Definition 1.16, we need to see if \( W \) is closed under vector addition and scalar multiplication. Let \( \vec{v}_1, \vec{v}_2 \in W \). Then
\[ \vec{v}_1 = [2x_1, x_1 + y_1, y_1] \] and \( \vec{v}_2 = [2x_2, x_2 + y_2, y_2] \) for some \( x_1, x_2, y_1, y_2 \in \mathbb{R} \).
Page 99 Number 8. Determine whether the set 
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\[ \vec{v}_1 = [2x_1, x_1 + y_1, y_1] \] and \( \vec{v}_2 = [2x_2, x_2 + y_2, y_2] \) for some \( x_1, x_2, y_1, y_2 \in \mathbb{R} \). For vector addition,

\[
\begin{align*}
\vec{v}_1 + \vec{v}_2 &= [2x_1, x_1 + y_1, y_1] + [2x_2, x_2 + y_2, y_2] \\
&= [2x_1 + 2x_2, (x_1 + y_1) + (x_2 + y_2), y_1 + y_2] \\
&= [2(x_1 + x_2), (x_1 + x_2) + (y_1 + y_2), (y_1 + y_2)] \\
&= [2x, x + y, y] \text{ where } x = x_1 + x_2 \text{ and } y = y_1 + y_2.
\end{align*}
\]

So \( \vec{v}_1 + \vec{v}_2 \in W \) and \( W \) is closed under vector addition.
Page 99 Number 8. Determine whether the set
\[ W = \{ [2x, x + y, y] \mid x, y \in \mathbb{R} \} \] is a subspace of \( \mathbb{R}^3 \).

**Solution.** By Definition 1.16, we need to see if \( W \) is closed under vector addition and scalar multiplication. Let \( \vec{v}_1, \vec{v}_2 \in W \). Then
\[ \vec{v}_1 = [2x_1, x_1 + y_1, y_1] \] and \( \vec{v}_2 = [2x_2, x_2 + y_2, y_2] \) for some \( x_1, x_2, y_1, y_2 \in \mathbb{R} \). For vector addition,

\[
\begin{align*}
\vec{v}_1 + \vec{v}_2 &= [2x_1, x_1 + y_1, y_1] + [2x_2, x_2 + y_2, y_2] \\
&= [2x_1 + 2x_2, (x_1 + y_1) + (x_2 + y_2), y_1 + y_2] \\
&= [2(x_1 + x_2), (x_1 + x_2) + (y_1 + y_2), (y_1 + y_2)] \\
&= [2x, x + y, y] \text{ where } x = x_1 + x_2 \text{ and } y = y_1 + y_2.
\end{align*}
\]

So \( \vec{v}_1 + \vec{v}_2 \in W \) and \( W \) is closed under vector addition.
Page 99 Number 8. Determine whether the set
\[ W = \{ [2x, x + y, y] \mid x, y \in \mathbb{R} \} \] is a subspace of \( \mathbb{R}^3 \).

Solution (continued). For scalar multiplication, let \( r \in \mathbb{R} \) and consider
\[
\begin{align*}
\mathbf{r}\mathbf{v}_1 &= r[2x_1, x_1 + y_1, y_1] = [r(2x_1), r(x_1 + y_1), r(y_1)] \\
&= [2(rx_1), (rx_1) + (ry_1), (ry_1)] \\
&= [2x, x + y, y] \text{ where } x = rx_1 \text{ and } y = ry_1.
\end{align*}
\]
So \( r\mathbf{v}_1 \in W \) and \( W \) is closed under scalar multiplication. Therefore, \( W \) is a subspace of \( \mathbb{R}^3 \). \( \square \)
Determine whether the set \( W = \{ [2x, x + y, y] \mid x, y \in \mathbb{R} \} \) is a subspace of \( \mathbb{R}^3 \).

**Solution (continued).** For scalar multiplication, let \( r \in \mathbb{R} \) and consider

\[
\begin{align*}
    r\vec{v}_1 &= r[2x_1, x_1 + y_1, y_1] = [r(2x_1), r(x_1 + y_1), r(y_1)] \\
    &= [2(rx_1), (rx_1) + (ry_1), (ry_1)] \\
    &= [2x, x + y, y] \text{ where } x = rx_1 \text{ and } y = ry_1.
\end{align*}
\]

So \( r\vec{v}_1 \in W \) and \( W \) is closed under scalar multiplication. Therefore, \( W \) is a subspace of \( \mathbb{R}^3 \). \( \square \)
Theorem 1.14. Subspace Property of a Span

Let \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) be the span of \( k > 0 \) vectors in \( \mathbb{R}^n \) Then \( W \) is a subspace of \( \mathbb{R}^n \). (The vectors \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \) are said to span or generate the subspace.)

Proof. We use Definition 1.16, “Closure and Subspace.” Let \( \vec{u}, \vec{v} \in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) and let \( c \) be a scalar.
Theorem 1.14. Subspace Property of a Span

Let $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ be the span of $k > 0$ vectors in $\mathbb{R}^n$. Then $W$ is a subspace of $\mathbb{R}^n$. (The vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are said to span or generate the subspace.)

Proof. We use Definition 1.16, “Closure and Subspace.” Let $\vec{u}, \vec{v} \in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ and let $c$ be a scalar. Then $\vec{u} = r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k$ and $\vec{v} = s_1\vec{w}_1 + s_2\vec{w}_2 + \cdots + s_k\vec{w}_k$ for some scalars $r_i$ and $s_i$. 
Theorem 1.14. Subspace Property of a Span

Let \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) be the span of \( k > 0 \) vectors in \( \mathbb{R}^n \). Then \( W \) is a subspace of \( \mathbb{R}^n \). (The vectors \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \) are said to span or generate the subspace.)

**Proof.** We use Definition 1.16, “Closure and Subspace.” Let \( \vec{u}, \vec{v} \in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) and let \( c \) be a scalar. Then

\[
\vec{u} = r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k \quad \text{and} \quad \vec{v} = s_1 \vec{w}_1 + s_2 \vec{w}_2 + \cdots + s_k \vec{w}_k
\]

for some scalars \( r_i \) and \( s_i \). Then

\[
\vec{u} + \vec{v} = (r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k) + (s_1 \vec{w}_1 + s_2 \vec{w}_2 + \cdots + s_k \vec{w}_k)
\]

by S1 and S2

\[
\in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)
\]

and so \( \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) is closed under vector addition.
Theorem 1.14. Subspace Property of a Span

Let $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ be the span of $k > 0$ vectors in $\mathbb{R}^n$. Then $W$ is a subspace of $\mathbb{R}^n$. (The vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are said to span or generate the subspace.)

**Proof.** We use Definition 1.16, “Closure and Subspace.” Let $\vec{u}, \vec{v} \in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ and let $c$ be a scalar. Then

$\vec{u} = r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k$ and $\vec{v} = s_1 \vec{w}_1 + s_2 \vec{w}_2 + \cdots + s_k \vec{w}_k$ for some scalars $r_i$ and $s_i$. Then

$$
\vec{u} + \vec{v} = (r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k) + (s_1 \vec{w}_1 + s_2 \vec{w}_2 + \cdots + s_k \vec{w}_k)
$$

$$
= (r_1 + s_1) \vec{w}_1 + (r_2 + s_2) \vec{w}_2 + \cdots + (r_k + s_k) \vec{w}_k \text{ by S1 and S2}
$$

$$
\in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)
$$

and so $\text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ is closed under vector addition.
Theorem 1.14. Subspace Property of a Span

Let $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ be the span of $k > 0$ vectors in $\mathbb{R}^n$. Then $W$ is a subspace of $\mathbb{R}^n$. (The vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are said to span or generate the subspace.)

Proof (continued). Next,

$$c\vec{u} = c(r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k)$$
$$= (cr_1)\vec{w}_1 + (cr_2)\vec{w}_2 + \cdots + (cr_k)\vec{w}_k \text{ by S1 and S3}$$
$$\in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$$

and so $\text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ is closed under scalar multiplication. So by Definition 1.16 $\text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ is a subspace of $\mathbb{R}^n$. \qed
Theorem 1.14. Subspace Property of a Span

Let $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ be the span of $k > 0$ vectors in $\mathbb{R}^n$. Then $W$ is a subspace of $\mathbb{R}^n$. (The vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are said to *span* or *generate* the subspace.)

**Proof (continued).** Next,

$$c\vec{u} = c(r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k)$$

$$= (cr_1)\vec{w}_1 + (cr_2)\vec{w}_2 + \cdots + (cr_k)\vec{w}_k \text{ by S1 and S3}$$

$$\in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$$

and so $\text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ is closed under scalar multiplication. So by Definition 1.16 $\text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ is a subspace of $\mathbb{R}^n$. □
Find a generating set for the solution set of the homogeneous linear system:

\[
\begin{align*}
    x_1 - x_2 + x_3 - x_4 &= 0 \\
    x_2 + x_3 &= 0 \\
    x_1 + 2x_2 - x_3 + 3x_4 &= 0.
\end{align*}
\]

**Solution.** We apply Gauss-Jordan elimination to the augmented matrix:

\[
\begin{bmatrix}
    1 & -1 & 1 & -1 & | & 0 \\
    0 & 1 & 1 & 0 & | & 0 \\
    1 & 2 & -1 & 3 & | & 0
\end{bmatrix}
\]

\[
R_3 \rightarrow R_3 - R_1
\]

\[
\begin{bmatrix}
    1 & -1 & 1 & -1 & | & 0 \\
    0 & 1 & 1 & 0 & | & 0 \\
    0 & 3 & -2 & 4 & | & 0
\end{bmatrix}
\]
Page 100 Number 18

Page 100 Number 18. Find a generating set for the solution set of the homogeneous linear system:

\[ \begin{align*}
    x_1 & - x_2 + x_3 - x_4 = 0 \\
    x_2 + x_3 &= 0 \\
    x_1 + 2x_2 - x_3 + 3x_4 &= 0.
\end{align*} \]

Solution. We apply Gauss-Jordan elimination to the augmented matrix:

\[
\begin{bmatrix}
    1 & -1 & 1 & -1 & 0 \\
    0 & 1 & 1 & 0 & 0 \\
    1 & 2 & -1 & 3 & 0
\end{bmatrix} \rightarrow 
\begin{bmatrix}
    1 & -1 & 1 & -1 & 0 \\
    0 & 1 & 1 & 0 & 0 \\
    0 & 3 & -2 & 4 & 0
\end{bmatrix} \rightarrow 
\begin{bmatrix}
    1 & 0 & 2 & -1 & 0 \\
    0 & 1 & 1 & 0 & 0 \\
    0 & 0 & -5 & 4 & 0
\end{bmatrix} \rightarrow 
\begin{bmatrix}
    1 & 0 & 2 & -1 & 0 \\
    0 & 1 & 1 & 0 & 0 \\
    0 & 0 & 1 & -4/5 & 0
\end{bmatrix}
\]
Page 100 Number 18

**Page 100 Number 18.** Find a generating set for the solution set of the homogeneous linear system:

\[
\begin{align*}
x_1 - x_2 + x_3 - x_4 &= 0 \\
x_2 + x_3 &= 0 \\
x_1 + 2x_2 - x_3 + 3x_4 &= 0.
\end{align*}
\]

**Solution.** We apply Gauss-Jordan elimination to the augmented matrix:

\[
\begin{bmatrix}
1 & -1 & 1 & -1 & | & 0 \\
0 & 1 & 1 & 0 & | & 0 \\
1 & 2 & -1 & 3 & | & 0
\end{bmatrix}
\]

\[
R_3 \rightarrow R_3 - R_1 \\
\begin{bmatrix}
1 & -1 & 1 & -1 & | & 0 \\
0 & 1 & 1 & 0 & | & 0 \\
0 & 3 & -2 & 4 & | & 0
\end{bmatrix}
\]

\[
R_1 \rightarrow R_1 + R_2 \\
\begin{bmatrix}
1 & 0 & 2 & -1 & | & 0 \\
0 & 1 & 1 & 0 & | & 0 \\
0 & 0 & -5 & 4 & | & 0
\end{bmatrix}
\]

\[
R_3 \rightarrow R_3 - 3R_2 \\
\begin{bmatrix}
1 & 0 & 2 & -1 & | & 0 \\
0 & 1 & 1 & 0 & | & 0 \\
0 & 0 & -5 & 4 & | & 0
\end{bmatrix}
\]

\[
R_3 \rightarrow R_3 / (-5) \Rightarrow R_3 / (-5) \\
\begin{bmatrix}
1 & 0 & 2 & -1 & | & 0 \\
0 & 1 & 1 & 0 & | & 0 \\
0 & 0 & 1 & -4/5 & | & 0
\end{bmatrix}
\]

...
Solution (continued). . .

\[
\begin{bmatrix}
1 & 0 & 2 & -1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & -4/5 & 0
\end{bmatrix}
\begin{array}{c}
R_1 \rightarrow R_1 - 2R_3 \\
R_2 \rightarrow R_2 - R_3
\end{array}
\begin{bmatrix}
1 & 0 & 0 & 3/5 & 0 \\
0 & 1 & 0 & 4/5 & 0 \\
0 & 0 & 1 & -4/5 & 0
\end{bmatrix}
\]

Returning to a system of equations,

\[
\begin{align*}
x_1 + \frac{3}{5}x_4 &= 0 & \text{or} & & x_1 &= -\frac{3}{5}x_4 \\
x_2 + \frac{4}{5}x_4 &= 0 & & x_2 &= -\frac{4}{5}x_4 \\
x_3 - \frac{4}{5}x_4 &= 0 & & x_3 &= \frac{4}{5}x_4 \\
x_4 &= x_4.
\end{align*}
\]
Solution (continued). . .

\[
\begin{bmatrix}
1 & 0 & 2 & -1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & -4/5
\end{bmatrix}
\begin{bmatrix}
R_1 \rightarrow R_1 - 2R_3 \\
R_2 \rightarrow R_2 - R_3
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 3/5 & 0 \\
0 & 1 & 4/5 & 0 \\
0 & 0 & 1 & -4/5
\end{bmatrix}
\]

Returning to a system of equations,

\[
\begin{align*}
\begin{align*}
x_1 + \left(\frac{3}{5}\right)x_4 &= 0 & \text{or} & \quad x_1 &= -\left(\frac{3}{5}\right)x_4 \\
x_2 + \left(\frac{4}{5}\right)x_4 &= 0 & \text{and} & \quad x_2 &= -\left(\frac{4}{5}\right)x_4 \\
x_3 - \left(\frac{4}{5}\right)x_4 &= 0 & \text{and} & \quad x_3 &= \left(\frac{4}{5}\right)x_4 \\
x_4 &= x_4.
\end{align*}
\end{align*}
\]

So let \(r = x_4\) be a free variable and we have that the general solution is of the form

\[
\vec{x} = \begin{bmatrix}
-3/5 \\
-4/5 \\
4/5 \\
1
\end{bmatrix} r \quad r \in \mathbb{R}
\]
Solution (continued). . .

\[
\begin{bmatrix}
1 & 0 & 2 & -1 & | & 0 \\
0 & 1 & 1 & 0 & | & 0 \\
0 & 0 & 1 & -4/5 & | & 0
\end{bmatrix}
\]

\[
\begin{align*}
R_1 & \rightarrow R_1 - 2R_3 \\
R_2 & \rightarrow R_2 - R_3
\end{align*}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 3/5 & | & 0 \\
0 & 1 & 0 & 4/5 & | & 0 \\
0 & 0 & 1 & -4/5 & | & 0
\end{bmatrix}
\]

Returning to a system of equations,

\[
\begin{align*}
x_1 + (3/5)x_4 &= 0 & \text{or} & & x_1 &= -(3/5)x_4 \\
x_2 + (4/5)x_4 &= 0 & & x_2 &= -(4/5)x_4 \\
x_3 - (4/5)x_4 &= 0 & & x_3 &= (4/5)x_4 \\
x_4 &= x_4.
\end{align*}
\]

So let \( r = x_4 \) be a free variable and we have that the general solution is of the form

\[
\vec{x} \in \left\{ r \begin{bmatrix} -3/5 \\ -4/5 \\ 4/5 \\ 1 \end{bmatrix} \mid r \in \mathbb{R} \right\}.
\]
Solution (continued). So a generating set for the system is

\[
\begin{bmatrix}
-3/5 \\
-4/5 \\
4/5 \\
4/5 \\
1
\end{bmatrix}
\].

Note: We could have let \( s = x_4/5 \) be a free variable in which case a generating set is given by the simpler

\[
\begin{bmatrix}
-3 \\
-4 \\
4 \\
5
\end{bmatrix}
\]. ∎
Solution (continued). So a generating set for the system is

\[
\begin{bmatrix}
-3/5 \\
-4/5 \\
4/5 \\
1
\end{bmatrix}.
\]

Note: We could have let \( s = x_4/5 \) be a free variable in which case a generating set is given by the simpler

\[
\begin{bmatrix}
-3 \\
-4 \\
4 \\
5
\end{bmatrix}.
\]
Theorem 1.15. Unique Linear Combinations.

The set \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\} is a basis for \(W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)\) if and only if

\[ r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = \vec{0} \]

implies

\[ r_1 = r_2 = \cdots = r_k = 0. \]

Proof. Suppose \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\} is a basis for \(W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)\). Then by Definition 1.17, “Basis for a Subspace,” every vector in \(W\) can be expressed uniquely as a linear combination of the \(\vec{w}_i\).
Theorem 1.15. Unique Linear Combinations.
The set \( \{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \} \) is a basis for \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) if and only if
\[
 r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k = \vec{0}
\]
implies
\[
 r_1 = r_2 = \cdots = r_k = 0.
\]

Proof. Suppose \( \{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \} \) is a basis for \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \). Then by Definition 1.17, “Basis for a Subspace,” every vector in \( W \) can be expressed uniquely as a linear combination of the \( \vec{w}_i \). In particular,
\[
 r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k = \vec{0}
\]
is satisfied for \( r_1 = r_2 = \cdots = r_k = 0 \) and the uniqueness condition implies that we must have \( r_1 = r_2 = \cdots = r_k = 0 \).
Theorem 1.15. Unique Linear Combinations.

The set \( \{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \} \) is a basis for \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) if and only if

\[
    r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k = \vec{0}
\]

implies

\[
    r_1 = r_2 = \cdots = r_k = 0.
\]

**Proof.** Suppose \( \{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \} \) is a basis for \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \). Then by Definition 1.17, “Basis for a Subspace,” every vector in \( W \) can be expressed uniquely as a linear combination of the \( \vec{w}_i \). In particular,

\[
    r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k = \vec{0}
\]

is satisfied for \( r_1 = r_2 = \cdots = r_k = 0 \) and the uniqueness condition implies that we must have \( r_1 = r_2 = \cdots = r_k = 0 \).
Theorem 1.15 (continued)

Theorem 1.15. Unique Linear Combinations.
The set \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\} is a basis for \(W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)\) if and only if
\[r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k = \vec{0}\]
implies
\[r_1 = r_2 = \cdots = r_k = 0.\]

Proof (continued). Now suppose that \(r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k = \vec{0}\)
implies that \(r_1 = r_2 = \cdots = r_k = 0\). Let \(\vec{w} \in W\) and suppose
\(\vec{w} = c_1 \vec{w}_1 + c_2 \vec{w}_2 + \cdots + c_k \vec{w}_k = d_1 \vec{w}_1 + d_2 \vec{w}_2 + \cdots + d_k \vec{w}_k\). Then
\(\vec{0} = \vec{w} - \vec{w} = (c_1 - d_1) \vec{w}_1 + (c_2 - d_2) \vec{w}_2 + \cdots + (c_k - d_k) \vec{w}_k\) (by S1 and S2). By hypothesis for this case, we must have \(c_1 - d_1 = c_2 - d_2 = \cdots = c_k - d_k = 0\).
Theorem 1.15. Unique Linear Combinations.
The set \( \{ \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k \} \) is a basis for \( W = \text{sp}(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k) \) if and only if
\[
    r_1 \mathbf{w}_1 + r_2 \mathbf{w}_2 + \cdots + r_k \mathbf{w}_k = \mathbf{0}
\]
implies
\[
    r_1 = r_2 = \cdots = r_k = 0.
\]

Proof (continued). Now suppose that \( r_1 \mathbf{w}_1 + r_2 \mathbf{w}_2 + \cdots + r_k \mathbf{w}_k = \mathbf{0} \) implies that \( r_1 = r_2 = \cdots = r_k = 0 \). Let \( \mathbf{w} \in W \) and suppose
\[
    \mathbf{w} = c_1 \mathbf{w}_1 + c_2 \mathbf{w}_2 + \cdots + c_k \mathbf{w}_k = d_1 \mathbf{w}_1 + d_2 \mathbf{w}_2 + \cdots + d_k \mathbf{w}_k.
\]
Then
\[
    \mathbf{0} = \mathbf{w} - \mathbf{w} = (c_1 - d_1) \mathbf{w}_1 + (c_2 - d_2) \mathbf{w}_2 + \cdots + (c_k - d_k) \mathbf{w}_k \quad \text{(by S1 and S2)}.
\]
By hypothesis for this case, we must have
\[
    c_1 - d_1 = c_2 - d_2 = \cdots = c_k - d_k = 0.
\]
That is, we must have \( c_1 = d_1, c_2 = d_2, \ldots, c_k = d_k \). Hence every vector of \( W \) is a unique linear combination of the \( \mathbf{w}_i \), as claimed.
Theorem 1.15. Unique Linear Combinations.
The set \( \{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \} \) is a basis for \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) if and only if
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\]
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\[
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Proof (continued). Now suppose that \( r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k = \vec{0} \) implies that \( r_1 = r_2 = \cdots = r_k = 0 \). Let \( \vec{w} \in W \) and suppose
\[
    \vec{w} = c_1 \vec{w}_1 + c_2 \vec{w}_2 + \cdots + c_k \vec{w}_k = d_1 \vec{w}_1 + d_2 \vec{w}_2 + \cdots + d_k \vec{w}_k.
\]
Then
\[
    \vec{0} = \vec{w} - \vec{w} = (c_1 - d_1) \vec{w}_1 + (c_2 - d_2) \vec{w}_2 + \cdots + (c_k - d_k) \vec{w}_k \quad \text{(by S1 and S2)}.
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By hypothesis for this case, we must have
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That is, we must have \( c_1 = d_1, c_2 = d_2, \ldots, c_k = d_k \). Hence every vector of \( W \) is a unique linear combination of the \( \vec{w}_i \), as claimed.
Page 100 Number 22(a). Use Theorem 1.15 to determine whether the set \([-1, 1], [1, 2]\) is a basis for the subspace of \(\mathbb{R}^2\) that it spans.

Solution. Based on Theorem 1.15, we consider scalars \(r_1, r_2 \in \mathbb{R}\) such that \(r_1[-1, 1] + r_2[1, 2] = [0, 0]\). This implies \([-r_1, r_1] + [r_2, 2r_2] = [0, 0]\) or \([-r_1 + r_2, r_1 + 2r_2] = [0, 0]\).
Use Theorem 1.15 to determine whether the set \([-1, 1], [1, 2]\) is a basis for the subspace of \(\mathbb{R}^2\) that it spans.

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\[
\begin{align*}
-r_1 + r_2 &= 0 \\
r_1 + 2r_2 &= 0.
\end{align*}
\]
Page 100 Number 22(a). Use Theorem 1.15 to determine whether the set \([-1, 1], [1, 2]\) is a basis for the subspace of \(\mathbb{R}^2\) that it spans.

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\[-r_1 + r_2 = 0 \quad (1)\]
\[r_1 + 2r_2 = 0. \quad (2)\]

From (1) we see that \(r_1 = r_2\) and so from (2) we need \(r_1 + 2(r_1) = 0\) or \(3r_1 = 0\) or \(r_1 = 0\). Since \(r_1 = r_2\) we also need \(r_2 = 0\).
Page 100 Number 22(a). Use Theorem 1.15 to determine whether the set \{\([-1, 1], [1, 2]\)\} is a basis for the subspace of \(\mathbb{R}^2\) that it spans.

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Page 100 Number 22(a). Use Theorem 1.15 to determine whether the set \{[−1, 1], [1, 2]\} is a basis for the subspace of \(\mathbb{R}^2\) that it spans.

Solution. Based on Theorem 1.15, we consider scalars \(r_1, r_2 \in \mathbb{R}\) such that \(r_1[−1, 1] + r_2[1, 2] = [0, 0]\). This implies \([-r_1, r_1] + [r_2, 2r_2] = [0, 0]\) or \([-r_1 + r_2, r_1 + 2r_2] = [0, 0]\). So we need

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Page 100 Number 22(b). Use Theorem 1.16 to determine whether the set \([-1, 1], [1, 2]\) is a basis for the subspace of \(\mathbb{R}^2\) that it spans.

**Solution.** We define matrix \(A\) which has as its *columns* the vectors in the set: 

\[
A = \begin{bmatrix}
-1 & 1 \\
1 & 2
\end{bmatrix}.
\]

By Theorem 1.16, we see that the columns of \(A\) form a basis for \(\mathbb{R}^2\) if and only if \(A\) is row equivalent to \(I\).

So 

\[
\begin{align*}
\text{R}_1 & \rightarrow \text{R}_2 \\
\text{R}_2 & \rightarrow \text{R}_2/3 \\
\text{R}_1 & \rightarrow \text{R}_1 - 2\text{R}_2 \\
\text{R}_2 & \rightarrow \text{R}_2
\end{align*}
\]

is equivalent to \(I\). So 

\[
A \sim I
\]

and hence the columns of \(A\) form a basis for \(\mathbb{R}^2\); that is, the set \([-1, 1], [1, 2]\) is a basis for the subspace of \(\mathbb{R}^2\) that it spans. (Since there are two vectors, their span is all of \(\mathbb{R}^2\).) □
Use Theorem 1.16 to determine whether the set \([-1, 1], [1, 2]\) is a basis for the subspace of \(\mathbb{R}^2\) that it spans.

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\[
A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 + R_1} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}.
\]
Use Theorem 1.16 to determine whether the set \{[-1, 1], [1, 2]\} is a basis for the subspace of \( \mathbb{R}^2 \) that it spans.

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1 & 2 \\
0 & 3
\end{bmatrix} \\
\xrightarrow{R_2 \rightarrow R_2/3} \begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - 2R_2} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = I.
\]

So \( A \sim I \) and hence the columns of \( A \) form a basis for \( \mathbb{R}^2 \); that is, the set \{[-1, 1], [1, 2]\} is a basis for the subspace of \( \mathbb{R}^2 \) that it spans.

(Since there are two vectors, their span is all of \( \mathbb{R}^2 \).)

\( \Box \)
Use Theorem 1.16 to determine whether the set \([-1, 1], [1, 2]\) is a basis for the subspace of \(\mathbb{R}^2\) that it spans.

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1 & 2 \\
0 & 3
\end{bmatrix} \xrightarrow{R_2 \rightarrow R_2/3} \begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - 2R_2} \begin{bmatrix}
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\]

So \(A \sim I\) and hence the columns of \(A\) form a basis for \(\mathbb{R}^2\); that is, the set \([-1, 1], [1, 2]\) is a basis for the subspace of \(\mathbb{R}^2\) that it spans. (Since there are two vectors, their span is all of \(\mathbb{R}^2\).) □
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A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 + R_1} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 / 3} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = I.
\]

So \(A \sim I\) and hence the columns of \(A\) form a basis for \(\mathbb{R}^2\); that is, the set \{[-1, 1], [1, 2]\} is a basis for the subspace of \(\mathbb{R}^2\) that it spans. (Since there are two vectors, their span is all of \(\mathbb{R}^2\).) \(\square\)
Example. Page 97 Example 6. A basis of \( \mathbb{R}^n \) cannot contain more than \( n \) vectors.

Proof. Suppose \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \) is a basis for \( \mathbb{R}^n \) and ASSUME \( k > n \). Consider the system \( A\vec{x} = \vec{0} \) where the column vectors of \( A \) are \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \). Then \( A \) has \( n \) rows and \( k \) columns (corresponding to \( n \) equations in \( k \) unknowns).
Example. Page 97 Example 6. A basis of $\mathbb{R}^n$ cannot contain more than $n$ vectors.

Proof. Suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a basis for $\mathbb{R}^n$ and ASSUME $k > n$. Consider the system $A\vec{x} = \vec{0}$ where the column vectors of $A$ are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$. Then $A$ has $n$ rows and $k$ columns (corresponding to $n$ equations in $k$ unknowns). With $n < k$, Corollary 2 implies there is a nontrivial solution to $A\vec{x} = \vec{0}$. But this corresponds to a linear combination of the columns of $A$ which equals $\vec{0}$ while not all the coefficients are 0. This CONTRADICTS Theorem 1.15 (since we then have two different linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ which equal $\vec{0}$).
Example. Page 97 Example 6. A basis of $\mathbb{R}^n$ cannot contain more than $n$ vectors.

Proof. Suppose \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \) is a basis for $\mathbb{R}^n$ and assume $k > n$. Consider the system $A\vec{x} = \vec{0}$ where the column vectors of $A$ are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$. Then $A$ has $n$ rows and $k$ columns (corresponding to $n$ equations in $k$ unknowns). With $n < k$, Corollary 2 implies there is a nontrivial solution to $A\vec{x} = \vec{0}$. But this corresponds to a linear combination of the columns of $A$ which equals $\vec{0}$ while not all the coefficients are 0. This CONTRADICTS Theorem 1.15 (since we then have two different linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ which equal $\vec{0}$). So the assumption that $k > n$ is false. Therefore, $k \leq n$. □
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Theorem 1.18

Theorem 1.18. Structure of the Solution Set of $A\vec{x} = \vec{b}$.

Let $A\vec{x} = \vec{b}$ be a linear system. If $\vec{p}$ is any particular solution of $A\vec{x} = \vec{b}$ and $\vec{h}$ is a solution to $A\vec{x} = \vec{0}$, then $\vec{p} + \vec{h}$ is a solution of $A\vec{x} = \vec{b}$. In fact, every solution of $A\vec{x} = \vec{b}$ has the form $\vec{p} + \vec{h}$ and the general solution is $\vec{x} = \vec{p} + \vec{h}$ where $A\vec{h} = \vec{0}$ (that is, $\vec{h}$ is an arbitrary element of the nullspace of $A$).

Proof. Let $\vec{p}$ be a particular solution of $A\vec{x} = \vec{b}$ (so that $A\vec{p} = \vec{b}$). Let $\vec{h}$ be a solution of the homogeneous system $A\vec{x} = \vec{0}$ (so that $A\vec{h} = \vec{0}$).
Theorem 1.18. Structure of the Solution Set of $A\vec{x} = \vec{b}$.

Let $A\vec{x} = \vec{b}$ be a linear system. If $\vec{p}$ is any particular solution of $A\vec{x} = \vec{b}$ and $\vec{h}$ is a solution to $A\vec{x} = \vec{0}$, then $\vec{p} + \vec{h}$ is a solution of $A\vec{x} = \vec{b}$. In fact, every solution of $A\vec{x} = \vec{b}$ has the form $\vec{p} + \vec{h}$ and the general solution is $\vec{x} = \vec{p} + \vec{h}$ where $A\vec{h} = \vec{0}$ (that is, $\vec{h}$ is an arbitrary element of the nullspace of $A$).

**Proof.** Let $\vec{p}$ be a particular solution of $A\vec{x} = \vec{b}$ (so that $A\vec{p} = \vec{b}$). Let $\vec{h}$ be a solution of the homogeneous system $A\vec{x} = \vec{0}$ (so that $A\vec{h} = \vec{0}$). Then

$$A(\vec{p} + \vec{h}) = A\vec{p} + A\vec{h} \text{ by Theorem 1.2.A(10),}$$

Distribution of Matrix Multiplication

$$= \vec{b} + \vec{0} = \vec{b}.$$ 

So $\vec{p} + \vec{h}$ is a solution of $A\vec{x} = \vec{b}$. 


Theorem 1.18. Structure of the Solution Set of $A\vec{x} = \vec{b}$.

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Theorem 1.18 (continued)

**Theorem 1.18. Structure of the Solution Set of** \( A\vec{x} = \vec{b} \).

Let \( A\vec{x} = \vec{b} \) be a linear system. If \( \vec{p} \) is any particular solution of \( A\vec{x} = \vec{b} \) and \( \vec{h} \) is a solution to \( A\vec{x} = \vec{0} \), then \( \vec{p} + \vec{h} \) is a solution of \( A\vec{x} = \vec{b} \). In fact, every solution of \( A\vec{x} = \vec{b} \) has the form \( \vec{x} = \vec{p} + \vec{h} \) where \( A\vec{h} = \vec{0} \) (that is, \( \vec{h} \) is an arbitrary element of the nullspace of \( A \)).

**Proof (continued).** Now suppose \( \vec{q} \) is any solution to \( A\vec{x} = \vec{b} \). With \( \vec{p} \) as a particular solution to \( A\vec{x} = \vec{b} \) we have

\[
A(\vec{q} - \vec{p}) = A\vec{q} - A\vec{p} \text{ by Theorem 1.2.A(10),}
\]

Distribution of Matrix Multiplication

\[
= \vec{b} - \vec{b} = \vec{0}.
\]
Theorem 1.18. Structure of the Solution Set of $A\vec{x} = \vec{b}$.

Let $A\vec{x} = \vec{b}$ be a linear system. If $\vec{p}$ is any particular solution of $A\vec{x} = \vec{b}$ and $\vec{h}$ is a solution to $A\vec{x} = \vec{0}$, then $\vec{p} + \vec{h}$ is a solution of $A\vec{x} = \vec{b}$. In fact, every solution of $A\vec{x} = \vec{b}$ has the form $\vec{p} + \vec{h}$ and the general solution is $\vec{x} = \vec{p} + \vec{h}$ where $A\vec{h} = \vec{0}$ (that is, $\vec{h}$ is an arbitrary element of the nullspace of $A$).

Proof (continued). Now suppose $\vec{q}$ is any solution to $A\vec{x} = \vec{b}$. With $\vec{p}$ as a particular solution to $A\vec{x} = \vec{b}$ we have

$$A(\vec{q} - \vec{p}) = A\vec{q} - A\vec{p} \text{ by Theorem 1.2.A(10),}$$

$$\text{Distribution of Matrix Multiplication}$$

$$= \vec{b} - \vec{b} = \vec{0}.$$ 

So $\vec{q} - \vec{p}$ is a solution of $A\vec{x} = \vec{0}$, say $\vec{q} - \vec{p} = \vec{h}$. So $\vec{q} = \vec{p} + \vec{h}$ and every solution $\vec{x}$ of $A\vec{x} = \vec{b}$ is of the form $\vec{p} + \vec{h}$ where $\vec{p}$ is a particular solution of $A\vec{x} + \vec{b}$ and $\vec{h}$ is any solution of $A\vec{x} = \vec{h}$.
Theorem 1.18 (continued)

**Theorem 1.18. Structure of the Solution Set of** \( A\vec{x} = \vec{b} \).

Let \( A\vec{x} = \vec{b} \) be a linear system. If \( \vec{p} \) is any particular solution of \( A\vec{x} = \vec{b} \) and \( \vec{h} \) is a solution to \( A\vec{x} = \vec{0} \), then \( \vec{p} + \vec{h} \) is a solution of \( A\vec{x} = \vec{b} \). In fact, every solution of \( A\vec{x} = \vec{b} \) has the form \( \vec{p} + \vec{h} \) and the general solution is \( \vec{x} = \vec{p} + \vec{h} \) where \( A\vec{h} = \vec{0} \) (that is, \( \vec{h} \) is an arbitrary element of the nullspace of \( A \)).

**Proof (continued).** Now suppose \( \vec{q} \) is any solution to \( A\vec{x} = \vec{b} \). With \( \vec{p} \) as a particular solution to \( A\vec{x} = \vec{b} \) we have

\[
A(\vec{q} - \vec{p}) = A\vec{q} - A\vec{p} \quad \text{by Theorem 1.2.A(10),}
\]

Distribution of Matrix Multiplication

\[
= \vec{b} - \vec{b} = \vec{0}.
\]

So \( \vec{q} - \vec{p} \) is a solution of \( A\vec{x} = \vec{0} \), say \( \vec{q} - \vec{p} = \vec{h} \). So \( \vec{q} = \vec{p} + \vec{h} \) and every solution \( \vec{x} \) of \( A\vec{x} = \vec{b} \) is of the form \( \vec{p} + \vec{h} \) where \( \vec{p} \) is a particular solution of \( A\vec{x} + \vec{b} \) and \( \vec{h} \) is any solution of \( A\vec{x} = \vec{h} \). \( \square \)
Solve the linear system

\[
\begin{align*}
x_1 - 2x_2 + x_3 + x_4 &= 4 \\
2x_1 + x_2 - 3x_3 - x_4 &= 6 \\
x_1 - 7x_2 - 6x_3 + 2x_4 &= 6
\end{align*}
\]

and express the solution set in a form that illustrates Theorem 1.18.

**Solution.** We apply Gauss-Jordan elimination to the augmented matrix:

\[
\begin{bmatrix}
1 & -2 & 1 & 1 & 4 \\
2 & 1 & -3 & -1 & 6 \\
1 & -7 & -6 & 2 & 6
\end{bmatrix}
\]

\[
R_2 \rightarrow R_2 - 2R_1 \\
R_3 \rightarrow R_3 - R_1
\]

\[
\begin{bmatrix}
1 & -2 & 1 & 1 & 4 \\
0 & 5 & -5 & -3 & -2 \\
0 & -5 & -7 & 1 & 2
\end{bmatrix}
\]

\[
R_3 \rightarrow R_3 + R_2
\]
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Solve the linear system

\[
\begin{align*}
    x_1 - 2x_2 + x_3 + x_4 &= 4 \\
    2x_1 + x_2 - 3x_3 - x_4 &= 6 \\
    x_1 - 7x_2 - 6x_3 + 2x_4 &= 6
\end{align*}
\]

and express the solution set in a form that illustrates Theorem 1.18.

Solution. We apply Gauss-Jordan elimination to the augmented matrix:

\[
\begin{bmatrix}
    1 & -2 & 1 & 1 & 4 \\
    2 & 1 & -3 & -1 & 6 \\
    1 & -7 & -6 & 2 & 6
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1 & -2 & 1 & 1 & 4 \\
    0 & 5 & -5 & -3 & -2 \\
    0 & -5 & -7 & 1 & 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1 & -2 & 1 & 1 & 4 \\
    0 & 5 & -5 & -3 & -2 \\
    0 & 0 & -12 & -2 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1 & -2 & 1 & 1 & 4 \\
    0 & 1 & -1 & -3/5 & -2/5 \\
    0 & 0 & 1 & 1/6 & 0
\end{bmatrix}
\]
Solve the linear system

\[
\begin{align*}
  x_1 & - 2x_2 + x_3 + x_4 = 4 \\
  2x_1 & + x_2 - 3x_3 - x_4 = 6 \\
  x_1 & - 7x_2 - 6x_3 + 2x_4 = 6
\end{align*}
\]

and express the solution set in a form that illustrates Theorem 1.18.

**Solution.** We apply Gauss-Jordan elimination to the augmented matrix:

\[
\begin{bmatrix}
  1 & -2 & 1 & 1 & | & 4 \\
  2 & 1 & -3 & -1 & | & 6 \\
  1 & -7 & -6 & 2 & | & 6 \\
\end{bmatrix}
\]

\[
\begin{array}{l}
  R_2 \rightarrow R_2 - 2R_1 \\
  R_3 \rightarrow R_3 - R_1 \\
\end{array}
\]

\[
\begin{bmatrix}
  1 & -2 & 1 & 1 & | & 4 \\
  0 & 5 & -5 & -3 & | & -2 \\
  0 & -5 & -7 & 1 & | & 2 \\
\end{bmatrix}
\]

\[
\begin{array}{l}
  R_2 \rightarrow R_2 / 5 \\
  R_3 \rightarrow R_3 / (-12) \\
\end{array}
\]

\[
\begin{bmatrix}
  1 & -2 & 1 & 1 & | & 4 \\
  0 & 1 & -1 & -3/5 & | & -2/5 \\
  0 & 0 & 1 & 1/6 & | & 0 \\
\end{bmatrix}
\]
Solution (continued).

\[
\begin{align*}
R_1 &\rightarrow R_1 + 2R_2 \quad \begin{bmatrix}
1 & 0 & -1 & -1/5 & 16/5 \\
0 & 1 & -1 & -3/5 & -2/5 \\
0 & 0 & 1 & 1/6 & 0
\end{bmatrix} \\
R_1 &\rightarrow R_1 + R_3 \\
R_2 &\rightarrow R_2 + R_3 \\
\end{align*}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & -1/30 & 16/5 \\
0 & 1 & 0 & -13/30 & -2/5 \\
0 & 0 & 1 & 1/6 & 0
\end{bmatrix}.
\]

This corresponds to the system of equations:

\[
\begin{align*}
x_1 - \frac{1}{30} x_4 &= \frac{16}{5} \\
x_2 - \frac{13}{30} x_4 &= -\frac{2}{5} \\
\end{align*}
\]

For a particular solution \( \vec{p} \) to the original system of equations we choose to set \( x_4 = 0 \) so that...
Solution (continued).

\[
\begin{align*}
R_1 \rightarrow R_1 + 2R_2 & \quad \begin{bmatrix} 1 & 0 & -1 & -1/5 & 16/5 \\ 0 & 1 & -1 & -3/5 & -2/5 \\ 0 & 0 & 1 & 1/6 & 0 \end{bmatrix} \\
R_1 \rightarrow R_1 + R_3 & \quad \begin{bmatrix} 1 & 0 & 0 & -1/30 & 16/5 \\ 0 & 1 & 0 & -13/30 & -2/5 \\ 0 & 0 & 1 & 1/6 & 0 \end{bmatrix}.
\end{align*}
\]

This corresponds to the system of equations:

\[
\begin{align*}
x_1 & - (1/30)x_4 = 16/5 \\
x_2 & - (13/30)x_4 = -2/5 \\
x_3 & + (1/6)x_4 = 0
\end{align*}
\]

For a particular solution \( \vec{p} \) to the original system of equations we choose to set \( x_4 = 0 \) so that...
Solution (continued).

\[
\begin{array}{ccc|c}
R_1 \rightarrow R_1 + 2R_2 & 1 & 0 & -1 & -1/5 & 16/5 \\
& 0 & 1 & -1 & -3/5 & -2/5 \\
& 0 & 0 & 1 & 1/6 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc|c}
R_1 \rightarrow R_1 + R_3 & 1 & 0 & 0 & -1/30 & 16/5 \\
R_2 \rightarrow R_2 + R_3 & 0 & 1 & 0 & -13/30 & -2/5 \\
& 0 & 0 & 1 & 1/6 & 0 \\
\end{array}
\]

This corresponds to the system of equations:

\[
\begin{aligned}
x_1 & - (1/30)x_4 = 16/5 \\
x_2 & - (13/30)x_4 = -2/5 \\
x_3 & + (1/6)x_4 = 0 \\
\end{aligned}
\]

For a particular solution \( \vec{p} \) to the original system of equations we choose to set \( x_4 = 0 \) so that...
Solution (continued). \[ \vec{p} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16/5 \\ -2/5 \\ 0 \\ 0 \end{bmatrix}. \]

With \( A \) as the coefficient matrix, the homogeneous system \( A\vec{x} = \vec{0} \) reduces to a similar system of equations but with only 0's on the right hand side:

\[
\begin{align*}
  x_1 - (1/30)x_4 &= 0 \quad \text{or} \quad x_1 = (1/30)x_4 \\
  x_2 - (13/30)x_4 &= 0 \quad \text{or} \quad x_2 = (13/30)x_4 \\
  x_3 + (1/6)x_4 &= 0 \quad \text{or} \quad x_3 = -(1/6)x_4 \\
  x_4 &= x_4
\end{align*}
\]
Solution (continued). \[ \vec{p} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16/5 \\ -2/5 \\ 0 \\ 0 \end{bmatrix}. \]

With \( A \) as the coefficient matrix, the homogeneous system \( A\vec{x} = \vec{0} \) reduces to a similar system of equations but with only 0’s on the right hand side:

\[
\begin{align*}
    x_1 &- \frac{1}{30} x_4 = 0 & \text{or} & \quad x_1 = \frac{1}{30} x_4 \\
    x_2 &- \frac{13}{30} x_4 = 0 & \quad x_2 = \frac{13}{30} x_4 \\
    x_3 &+ \frac{1}{6} x_4 = 0 & \quad x_3 = -\frac{1}{6} x_4 \\
    x_4 &= x_4
\end{align*}
\]

So \( x_4 \) acts as a free variable in the associated homogeneous system of equations. To simplify the numbers, we set \( x_4 = 30r \) where \( r \in \mathbb{R} \) (since \( r \) is any element of \( \mathbb{R} \) then \( 30r \) is any element of \( \mathbb{R} \), and conversely).
Solution (continued). \( \vec{p} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16/5 \\ -2/5 \\ 0 \\ 0 \end{bmatrix} \). With \( A \) as the coefficient matrix, the homogeneous system \( A\vec{x} = \vec{0} \) reduces to a similar system of equations but with only 0’s on the right hand side:

\[
\begin{align*}
x_1 & - \frac{1}{30} x_4 = 0 \quad \text{or} \quad x_1 = \frac{1}{30} x_4 \\
x_2 & - \frac{13}{30} x_4 = 0 \quad x_2 = \frac{13}{30} x_4 \\
x_3 & + \frac{1}{6} x_4 = 0 \quad x_3 = -\frac{1}{6} x_4 \\
x_4 & = x_4
\end{align*}
\]

So \( x_4 \) acts as a free variable in the associated homogeneous system of equations. To simplify the numbers, we set \( x_4 = 30r \) where \( r \in \mathbb{R} \) (since \( r \) is any element of \( \mathbb{R} \) then \( 30r \) is any element of \( \mathbb{R} \), and conversely).
Solution (continued). This gives the solution to the homogeneous system as
\[ x_1 = (1/30)(30r) = r \]
\[ x_2 = (13/30)(30r) = 13r \]
\[ x_3 = -(1/6)(30r) = -5r \]
\[ x_4 = 30r. \]
So the solution set to the homogeneous system of equations \( A\vec{x} = \vec{b} \) is
\[
\begin{pmatrix}
1 \\
13 \\
-5 \\
30
\end{pmatrix} r \quad r \in \mathbb{R}
\]
(this is the nullspace of \( A \)).
Solution (continued). This gives the solution to the homogeneous system as

\[ x_1 = \frac{1}{30}(30r) = r \]
\[ x_2 = \frac{13}{30}(30r) = 13r \]
\[ x_3 = -\frac{1}{6}(30r) = -\frac{5}{3}r \]
\[ x_4 = 30r. \]

So the solution set to the homogeneous system of equations \( A\vec{x} = \vec{b} \) is

\[
\begin{bmatrix}
1 \\
13 \\
-5 \\
30
\end{bmatrix}
\begin{bmatrix}
r \\
r \\
r \\
r
\end{bmatrix}
\quad r \in \mathbb{R}
\]

(this is the nullspace of \( A \)). Therefore, in the notation of Theorem 1.18, the general solutions to the original (nonhomogeneous) system of equations is

\[ \vec{x} = \vec{p} + \vec{h} \]

where \( \vec{p} = \begin{bmatrix} 16/5 \\ -2/5 \\ 0 \\ 0 \end{bmatrix} \) and \( \vec{h} \in \begin{bmatrix} r \\ 13 \\ -5 \\ 30 \end{bmatrix} \) \( r \in \mathbb{R} \). \( \square \)
Solution (continued). This gives the solution to the homogeneous system as

\[ x_1 = (1/30)(30r) = r \]
\[ x_2 = (13/30)(30r) = 13r \]
\[ x_3 = -(1/6)(30r) = -5r \]
\[ x_4 = 30r. \]

So the solution set to the homogeneous system of equations \( A\vec{x} = \vec{b} \) is

\[
\begin{pmatrix}
 1 \\
 13 \\
 -5 \\
 30
\end{pmatrix} r 
\]

\( r \in \mathbb{R} \) (this is the nullspace of \( A \)). Therefore, in the notation of Theorem 1.18, the general solutions to the original (nonhomogeneous) system of equations is

\[
\vec{x} = \vec{p} + \vec{h} \quad \text{where} \quad \vec{p} =
\begin{bmatrix}
 16/5 \\
 -2/5 \\
 0 \\
 0
\end{bmatrix}
\quad \text{and} \quad \vec{h} \in \left\{ \begin{pmatrix}
 1 \\
 13 \\
 -5 \\
 30
\end{pmatrix} r 
\right\} \quad r \in \mathbb{R} \}.
\]

\( \square \)
Use Theorem 1.18 to prove why no system of linear equations can have exactly two solutions.

**Proof.** ASSUME to the contrary that linear system $A\vec{x} = \vec{b}$ does have exactly two solutions, say $\vec{p}_1$ and $\vec{p}_2$ (where $\vec{p}_1 \neq \vec{p}_2$).
Use Theorem 1.18 to prove why no system of linear equations can have exactly two solutions.

**Proof.** ASSUME to the contrary that linear system $A\vec{x} = \vec{b}$ does have exactly two solutions, say $\vec{p}_1$ and $\vec{p}_2$ (where $\vec{p}_1 \neq \vec{p}_2$). Then $A(\vec{p}_1 - \vec{p}_2) = A\vec{p}_1 - A\vec{p}_2 = \vec{b} - \vec{b} = \vec{0}$ and so $\vec{h} = \vec{p}_1 - \vec{p}_2 \neq \vec{0}$ is a solution to the homogeneous system $A\vec{x} = \vec{0}$. Now $\vec{p}_1 + \vec{h} = \vec{p}_1 + (\vec{p}_1 - \vec{p}_2) = 2\vec{p}_1 - \vec{p}_2 \neq \vec{p}_1$ (since $2\vec{p}_1 - \vec{p}_2 = \vec{p}_1$ implies $\vec{p}_1 = \vec{p}_2$, a contradiction) and $\vec{p}_1 + \vec{h} = \vec{p}_1 + (\vec{p}_1 - \vec{p}_2) = 2\vec{p}_1 - \vec{p}_2 \neq \vec{p}_2$ (since $2\vec{p}_1 - \vec{p}_2 = \vec{p}_2$ implies $\vec{p}_1 = \vec{p}_2$, a contradiction).
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**Page 101 Number 43.** Use Theorem 1.18 to prove why no system of linear equations can have exactly two solutions.

**Proof.** ASSUME to the contrary that linear system $A\vec{x} = \vec{b}$ does have exactly two solutions, say $\vec{p}_1$ and $\vec{p}_2$ (where $\vec{p}_1 \neq \vec{p}_2$). Then

$$A(\vec{p}_1 - \vec{p}_2) = A\vec{p}_1 - A\vec{p}_2 = \vec{b} - \vec{b} = \vec{0}$$

and so $\vec{h} = \vec{p}_1 - \vec{p}_2 \neq \vec{0}$ is a solution to the homogeneous system $A\vec{x} = \vec{0}$. Now

$$\vec{p}_1 + \vec{h} = \vec{p}_1 + (\vec{p}_1 - \vec{p}_2) = 2\vec{p}_1 - \vec{p}_2 \neq \vec{p}_1$$

(since $2\vec{p}_1 - \vec{p}_2 = \vec{p}_1$ implies $\vec{p}_1 = \vec{p}_2$, a contradiction) and

$$\vec{p}_1 + \vec{h} = \vec{p}_1 + (\vec{p}_1 - \vec{p}_2) = 2\vec{p}_1 - \vec{p}_2 \neq \vec{p}_2$$

(since $2\vec{p}_1 - \vec{p}_2 = \vec{p}_2$ implies $\vec{p}_1 = \vec{p}_2$, a contradiction). So by Theorem 1.18, $\vec{p}_1 + \vec{h}$ is a third solution to $A\vec{x} = \vec{b}$. This is a CONTRADICTION to the hypotheses.
Use Theorem 1.18 to prove why no system of linear equations can have exactly two solutions.

**Proof.** ASSUME to the contrary that linear system $A\vec{x} = \vec{b}$ does have exactly two solutions, say $\vec{p}_1$ and $\vec{p}_2$ (where $\vec{p}_1 \neq \vec{p}_2$). Then $A(\vec{p}_1 - \vec{p}_2) = A\vec{p}_1 - A\vec{p}_2 = \vec{b} - \vec{b} = \vec{0}$ and so $\vec{h} = \vec{p}_1 - \vec{p}_2 \neq \vec{0}$ is a solution to the homogeneous system $A\vec{x} = \vec{0}$. Now $\vec{p}_1 + \vec{h} = \vec{p}_1 + (\vec{p}_1 - \vec{p}_2) = 2\vec{p}_1 - \vec{p}_2 \neq \vec{p}_1$ (since $2\vec{p}_1 - \vec{p}_2 = \vec{p}_1$ implies $\vec{p}_1 = \vec{p}_2$, a contradiction) and $\vec{p}_1 + \vec{h} = \vec{p}_1 + (\vec{p}_1 - \vec{p}_2) = 2\vec{p}_1 - \vec{p}_2 \neq \vec{p}_2$ (since $2\vec{p}_1 - \vec{p}_2 = \vec{p}_2$ implies $\vec{p}_1 = \vec{p}_2$, a contradiction). So by Theorem 1.18, $\vec{p}_1 + \vec{h}$ is a third solution to $A\vec{x} = \vec{b}$. This is a CONTRADICTION to the hypotheses. So the assumption that $A\vec{x} = \vec{b}$ has exactly two solutions is false and the claim follows.
Use Theorem 1.18 to prove why no system of linear equations can have exactly two solutions.

**Proof.** ASSUME to the contrary that linear system $A\vec{x} = \vec{b}$ does have exactly two solutions, say $\vec{p}_1$ and $\vec{p}_2$ (where $\vec{p}_1 \neq \vec{p}_2$). Then $A(\vec{p}_1 - \vec{p}_2) = A\vec{p}_1 - A\vec{p}_2 = \vec{b} - \vec{b} = \vec{0}$ and so $\vec{h} = \vec{p}_1 - \vec{p}_2 \neq \vec{0}$ is a solution to the homogeneous system $A\vec{x} = \vec{0}$. Now $\vec{p}_1 + \vec{h} = \vec{p}_1 + (\vec{p}_1 - \vec{p}_2) = 2\vec{p}_1 - \vec{p}_2 \neq \vec{p}_1$ (since $2\vec{p}_1 - \vec{p}_2 = \vec{p}_1$ implies $\vec{p}_1 = \vec{p}_2$, a contradiction) and $\vec{p}_1 + \vec{h} = \vec{p}_1 + (\vec{p}_1 - \vec{p}_2) = 2\vec{p}_1 - \vec{p}_2 \neq \vec{p}_2$ (since $2\vec{p}_1 - \vec{p}_2 = \vec{p}_2$ implies $\vec{p}_1 = \vec{p}_2$, a contradiction). So by Theorem 1.18, $\vec{p}_1 + \vec{h}$ is a third solution to $A\vec{x} = \vec{b}$. This is a CONTRADICTION to the hypotheses. So the assumption that $A\vec{x} = \vec{b}$ has exactly two solutions is false and the claim follows. □
Let $W_1$ and $W_2$ be two subspaces of $\mathbb{R}^n$. Prove that their intersection $W_1 \cap W_2$ is also a subspace of $\mathbb{R}^n$.

**Proof.** We use Definition 1.16, “Closure and Subspace.” Let $\vec{u}, \vec{v} \in W_1 \cap W_2$. 

Since $W_1$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_1$. Since $W_2$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_2$. Hence $\vec{u} + \vec{v}$ is in both $W_1$ and $W_2$; that is, $\vec{u} + \vec{v} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under vector addition.

Now let $r$ be a scalar. Since $W_1$ is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r \vec{u} \in W_1$. Since $W_2$ is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r \vec{u} \in W_2$. Hence $r \vec{u}$ is in both $W_1$ and $W_2$; that is, $r \vec{u} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under scalar multiplication. By Definition 1.16, $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$. 

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**Page 101 Number 47.** Let $W_1$ and $W_2$ be two subspaces of $\mathbb{R}^n$. Prove that their intersection $W_1 \cap W_2$ is also a subspace of $\mathbb{R}^n$.

**Proof.** We use Definition 1.16, “Closure and Subspace.” Let $\vec{u}, \vec{v} \in W_1 \cap W_2$. Since $W_1$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_1$. Since $W_2$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_2$. 


Page 101 Number 47. Let $W_1$ and $W_2$ be two subspaces of $\mathbb{R}^n$. Prove that their intersection $W_1 \cap W_2$ is also a subspace of $\mathbb{R}^n$.

**Proof.** We use Definition 1.16, “Closure and Subspace.” Let $\vec{u}, \vec{v} \in W_1 \cap W_2$. Since $W_1$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_1$. Since $W_2$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_2$. Hence $\vec{u} + \vec{v}$ is in both $W_1$ and $W_2$; that is, $\vec{u} + \vec{v} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under vector addition.
Let $W_1$ and $W_2$ be two subspaces of $\mathbb{R}^n$. Prove that their intersection $W_1 \cap W_2$ is also a subspace of $\mathbb{R}^n$.

**Proof.** We use Definition 1.16, “Closure and Subspace.” Let $\vec{u}, \vec{v} \in W_1 \cap W_2$. Since $W_1$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_1$. Since $W_2$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_2$. Hence $\vec{u} + \vec{v}$ is in both $W_1$ and $W_2$; that is, $\vec{u} + \vec{v} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under vector addition.

Now let $r$ be a scalar. Since $W_1$ is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_1$. Since $W_2$ is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_2$. 
Let $W_1$ and $W_2$ be two subspaces of $\mathbb{R}^n$. Prove that their intersection $W_1 \cap W_2$ is also a subspace of $\mathbb{R}^n$.

**Proof.** We use Definition 1.16, “Closure and Subspace.” Let $\vec{u}, \vec{v} \in W_1 \cap W_2$. Since $W_1$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_1$. Since $W_2$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_2$. Hence $\vec{u} + \vec{v}$ is in both $W_1$ and $W_2$; that is, $\vec{u} + \vec{v} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under vector addition.

Now let $r$ be a scalar. Since $W_1$ is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_1$. Since $W_2$ is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_2$. Hence $r\vec{u}$ is in both $W_1$ and $W_2$; that is, $r\vec{u} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under scalar multiplication. By Definition 1.16, $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$. 
Page 101 Number 47. Let $W_1$ and $W_2$ be two subspaces of $\mathbb{R}^n$. Prove that their intersection $W_1 \cap W_2$ is also a subspace of $\mathbb{R}^n$.

**Proof.** We use Definition 1.16, “Closure and Subspace.” Let $\vec{u}, \vec{v} \in W_1 \cap W_2$. Since $W_1$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_1$. Since $W_2$ is a subspace then it is closed under vector addition (Definition 1.16) and so $\vec{u} + \vec{v} \in W_2$. Hence $\vec{u} + \vec{v}$ is in both $W_1$ and $W_2$; that is, $\vec{u} + \vec{v} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under vector addition.

Now let $r$ be a scalar. Since $W_1$ is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_1$. Since $W_2$ is a subspace then it is closed under scalar multiplication (Definition 1.16) and so $r\vec{u} \in W_2$. Hence $r\vec{u}$ is in both $W_1$ and $W_2$; that is, $r\vec{u} \in W_1 \cap W_2$. So $W_1 \cap W_2$ is closed under scalar multiplication. By Definition 1.16, $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$. \qed