Chapter 10: Solving Large Systems
Section 10.2 The $LU$-Factorization—Proofs of Theorems

Theorem 10.1. Unique Factorization.
Let $A$ be an $n \times n$ matrix. When a factorization $A = LDU$ exists where

1. $L$ is lower triangular with all main diagonal entries 1,
2. $U$ is upper triangular with all main diagonal entries 1, and
3. $D$ is a diagonal matrix with all main diagonal entries nonzero,

it is unique.

Proof. Suppose that $A = L_1 D_1 U_1 = L_2 D_2 U_2$ are two such factorizations. Then $L_1^{-1}$ and $L_2^{-1}$ are also lower triangular, $D_1^{-1}$ and $D_2^{-1}$ are both diagonal and $U_1^{-1}$ and $U_2^{-1}$ are both upper triangular. Since the diagonal entries of $L_1$, $L_2$, $U_1$, $U_2$ are all 1 then the diagonal entries of $L_1^{-1}$, $L_2^{-1}$, $U_1^{-1}$, $U_2^{-1}$ are also all 1.
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Proof (continued). We have $L_2^{-1}L_1 = D_2U_2U_1^{-1}D_1^{-1}$. A product of upper/lower triangular matrices is upper/lower triangular, so $L_2^{-1}L_1$ is lower triangular and $D_2U_2U_1^{-1}D_1^{-1}$ is upper triangular. Since $L_2^{-1}L_1 = D_2U_2U_1^{-1}D_1^{-1}$, then both sides of this equation must be the identity. So $L_2^{-1}L_1 = I$ and $L_1 = L_2$. Similarly, we can conclude $U_1U_2^{-1} = D_1^{-1}L_1^{-1}L_2D_2$ and both sides must be the identity. So $U_1U_2^{-1} = D_2$. We then have $L_1D_1U_1 = L_1D_2U_1$ and since all matrices are invertible, we conclude $D_1 = D_2$. We therefore have $L_1 = L_2$, $U_1 = U_2$, and $D_1 = D_2$. So the factorization of $A$ is unique. \qed