Chapter 10: Solving Large Systems
Section 10.2 The $LU$-Factorization—Proofs of Theorems
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**Theorem 10.A.** If $A$ is an $n \times n$ matrix which can be put in row echelon form without interchanging rows then there is a lower triangular matrix $L$ and an upper triangular matrix $U$ such that $A = LU$.

**Proof.** As described in the previous note, there is a sequence of $n \times n$ elementary matrices $E_i$ such that $E_hE_{h-1}\cdots E_2E_1A = U$ where each $E_i$ is an elementary matrix associated with the elementary row operation of row addition. Since $U$ is upper triangular then the row operations need only involve adding a multiple of one row to a lower row ($R_p \rightarrow R_p + sR_q$ where $p > q$).
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Theorem 10.A (continued)

**Theorem 10.A.** If $A$ is an $n \times n$ matrix which can be put in row echelon form without interchanging rows then there is a lower triangular matrix $L$ and an upper triangular matrix $U$ such that $A = LU$.

**Proof (continued).** Since $E_hE_{h-1} \cdots E_2E_1A = U$, then $A = E_1^{-1}E_2^{-1} \cdots E_{h-1}^{-1}E_h^{-1}U$. The product of square lower triangular matrices is lower triangular (this follows from the definition of matrix product; see Theorem 3.2.1(4) of my online notes for Theory of Matrices [MATH 5090] on Section 3.2. Multiplication of Matrices and Multiplication of Vectors and Matrices), so set $L = E_1^{-1}E_2^{-1} \cdots E_{h-1}^{-1}E_h^{-1}$. Then $L$ is lower triangular and $A = LU$, as claimed. □
Theorem 10.1. Unique Factorization.
Let $A$ be an $n \times n$ matrix. When a factorization $A = LDU$ exists where

1. $L$ is lower triangular with all main diagonal entries 1,
2. $U$ is upper triangular with all main diagonal entries 1, and
3. $D$ is a diagonal matrix with all main diagonal entries nonzero,

it is unique.

Proof. Suppose that $A = L_1 D_1 U_1 = L_2 D_2 U_2$ are two such factorizations. Then $L_1^{-1}$ and $L_2^{-1}$ are also lower triangular, $D_1^{-1}$ and $D_2^{-1}$ are both diagonal and $U_1^{-1}$ and $U_2^{-1}$ are both upper triangular. Since the diagonal entries of $L_1, L_2, U_1, U_2$ are all 1 then the diagonal entries of $L_1^{-1}, L_2^{-1}, U_1^{-1}, U_2^{-1}$ are also all 1.
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**Proof (continued).** We have $L_2^{-1}L_1 = D_2U_2U_1^{-1}D_1^{-1}$. A product of upper/lower triangular matrices is upper/lower triangular, so $L_2^{-1}L_1$ is lower triangular and $D_2U_2U_1^{-1}D_1^{-1}$ is upper triangular. Since $L_2^{-1}L_1 = D_2U_2U_1^{-1}D_1^{-1}$ then both sides of this equation must be the identity. So $L_2^{-1}L_1 = I$ and $L_1 = L_2$. 
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