Theorem 2.1

Theorem 2.1. Alternative Characterization of Basis

Let $W$ be a subspace of $\mathbb{R}^n$. A subset $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ of $W$ is a basis for $W$ if and only if

1. $W = \text{span}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ and
2. the vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ are linearly independent.

Proof. Recall that we defined $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ as a basis for $W$ if every vector in $W$ can be expressed as a unique linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ (see Definition 1.17).

Let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ be a basis for $W$. Then every vector in $W$ is a (unique) linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ and so these vectors span $W$ and (1) holds. To show linear independence, we consider the equation $r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = 0$. We know that $r_1 = r_2 = \cdots = r_k = 0$ is one possible choice for the $r_i$; but since $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a basis for $W$ then this is the only choice for the $r_i$ since $0$ is a unique linear combination of the $\vec{w}_i$.

Proof (continued). That is, $r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = 0$ implies $r_1 = r_2 = \cdots = r_k = 0$. So, by Definition 2.1, “Linear Dependence and Independence,” the $\vec{w}_i$ are not linearly dependent. That is, $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ are linearly independent and (2) holds.

Now suppose (1) and (2) hold. Then every vector in $W$ can be expressed as some linear combination of the $\vec{w}_i$ since the $\vec{w}_i$ span $W$ by (1). To show uniqueness of the linear combinations, suppose $\vec{v} \in W$ and $\vec{v} = r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = s_1\vec{w}_1 + s_2\vec{w}_2 + \cdots + s_k\vec{w}_k$. Then

\[(r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k) - (s_1\vec{w}_1 + s_2\vec{w}_2 + \cdots + s_k\vec{w}_k) = 0 \quad \text{and} \quad (r_1 - s_1)\vec{w}_1 + (r_2 - s_2)\vec{w}_2 + \cdots + (r_k - s_k)\vec{w}_k = 0.
\]

Since the $\vec{w}_i$ are linearly independent by (2), then $r_1 - s_1 = r_2 - s_2 = \cdots = r_k - s_k = 0$ by Note 2.1.A and so $r_1 = s_1, r_2 = s_2, \ldots, r_k = s_k$. That is, there is a unique linear combination of the $\vec{w}_i$ which equals $\vec{v}$. Since $\vec{v}$ is an arbitrary vector in $W$, then $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a basis for $W$. $\square$
Page 134 Number 10

**Page 134 Number 10.** Use Theorem 2.1.A, “Finding a Basis for \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \),” to find a basis for \( W = \text{sp}([-2, 3, 1], [3, -1, 2], [1, 2, 3], [-1, 5, 4]) \) in \( \mathbb{R}^3 \).

**Solution.** We create matrix \( A \) with the vectors in the spanning set as columns: \( A = \begin{bmatrix} -2 & 3 & 1 & -1 \\ 3 & -1 & 2 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix} \). Now we row reduce \( A \):

\[
A = \begin{bmatrix} -2 & 3 & 1 & -1 \\ 3 & -1 & 2 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 3 & 1 & -1 \\ 3 & -1 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -7 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = H.
\]

Since \( H \) is in row echelon form and has a pivot in each of the first two columns then, by Theorem 2.1.A, a set consisting of the first two vectors, \( \vec{w}_1, \vec{w}_2 \) is a basis for \( W \); that is, \([[-2, 3, 1], [3, -1, 2]]\) is a basis for \( W \).

Notice that the third vector is a linear combination of these two, \([1[-2, 3, 1] + 1[3, -1, 2]] = [1, 2, 3] \), and the fourth vector is a linear combination of these two, \([2[-2, 3, 1] + 1[3, -1, 2]] = [-1, 5, 4] \). \( \square \)

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**Page 135 Number 22.** Determine whether the set \([[1, -3, 2], [2, -5, 3], [4, 0, 1]]\) is linearly dependent or independent.

**Solution.** We use Theorem 2.1.A, “Finding a Basis for \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \),” and test to see if the set of vectors is a basis for its span. Let \( W = \text{sp}([1, -3, 2], [2, -5, 3], [4, 0, 1]) \). By Theorem 2.1, a basis for a vector space \( W \) is a linearly independent spanning set. Of course the set of vectors spans its span(!), so it is a basis of its span if and only if the set is a linearly independent set of vectors. We create matrix \( A \) with the vectors in the set as its columns and row reduce:

\[
A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & -5 & 0 \\ 2 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & 0 \end{bmatrix}
\]

\( \ldots \)

\[
A \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 12 \\ 0 & -1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 12 \\ 0 & 0 & 5 \end{bmatrix} = H.
\]

Since \( H \) is in row echelon form and has a pivot in each column then by Theorem 2.1.A the set of all three vectors in \([1, -3, 2], [2, -5, 3], [4, 0, 1]\) form a basis for \( W \). Therefore the set of vectors is linearly independent. \( \square \)
Theorem 2.2. Relative Sizes of Spanning and Independent Sets.

Let $W$ be a subspace of $\mathbb{R}^n$. Let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ be vectors in $W$ that span $W$ and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be vectors in $W$ that are independent. Then $k \geq m$.

**Proof.** We give a proof by contradiction. ASSUME $k < m$. Since the vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ span $W$ and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are in $W$ then there are scalars $a_{ij}$ such that:

\[
\begin{align*}
\vec{v}_1 &= a_{11}\vec{w}_1 + a_{21}\vec{w}_2 + \cdots + a_{k1}\vec{w}_k \\
\vec{v}_2 &= a_{12}\vec{w}_1 + a_{22}\vec{w}_2 + \cdots + a_{k2}\vec{w}_k \\
&\vdots \\
\vec{v}_m &= a_{1m}\vec{w}_1 + a_{2m}\vec{w}_2 + \cdots + a_{km}\vec{w}_k
\end{align*}
\]

Now summing these equation we get:

\[
x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m)\vec{w}_1 + (a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m)\vec{w}_2 + \cdots + (a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{km}x_m)\vec{w}_k.
\]

Consider the system of equations (which results by requiring each coefficient of the $\vec{w}_i$'s to be 0): ...

Proof (continued). ...\]

Corollary 2.1.A. Invariance of Dimension.

Any two bases of a subspace of $\mathbb{R}^n$ contains the same number of vectors.

**Proof.** Suppose that both $B$, a set of $k$ vectors, and $B'$, a set of $m$ vectors, are bases for $W$. Then both $B$ and $B'$ are linearly independent spanning sets of $W$ by Theorem 2.1, “Alternative Characterization of a Basis.” This means that $B$ is a set of $k$ vectors spanning $W$ and $B'$ is a set of $m$ linearly independent vectors in $W$. So by Theorem 2.2, “Relative Sizes of Spanning and Independent Sets,” $k \geq m$. But also $B'$ is a set of $m$ vectors spanning $W$ and $B$ is a set of $k$ linearly independent vectors in $W$. So by Theorem 2.2, $m \geq k$. Therefore $k = m$ and the bases $B$ and $B'$ have the same number of vectors. Since $B$ and $B'$ are arbitrary bases of $W$, the result follows. \[\square\]
Theorem 2.3. Existence and Determination of Bases.

(1) Every subspace \( W \neq \{0\} \) of \( \mathbb{R}^n \) has a basis and \( \dim(W) \leq n \).

Proof. Let \( W \) be a subspace of \( \mathbb{R}^n \) where \( W \neq \{0\} \). Then there is some \( \tilde{w}_1 \in W \) such that \( \tilde{w}_1 \neq 0 \). Set \( B_1 = \{\tilde{w}_1\} \). If \( W = \text{sp}(\tilde{w}_1) \) then stop, otherwise there is \( \tilde{w}_2 \in W \) such that \( \tilde{w}_2 \notin \text{sp}(\tilde{w}_1) \). Set \( B_2 = \{\tilde{w}_1, \tilde{w}_2\} \). Notice that \( \tilde{w}_1 \) and \( \tilde{w}_2 \) are linearly independent since \( \tilde{r}_1 \tilde{w}_1 + \tilde{r}_2 \tilde{w}_2 = 0 \) for \( \tilde{r}_1 \neq 0 \) implies \( \tilde{w}_1 = (-\tilde{r}_2/\tilde{r}_1) \tilde{w}_2 \), contradicting the choice of \( \tilde{w}_2 \notin \text{sp}(\tilde{w}_1) \) (and similarly if \( \tilde{r}_2 \neq 0 \)). If \( W = \text{sp}(\tilde{w}_1, \tilde{w}_2) \) then stop. Otherwise, continue inductively so that if \( W \neq \text{sp}(\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_i) \) where \( \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_i \) are linearly independent, then there is \( \tilde{w}_{i+1} \in W \) such that \( \tilde{w}_{i+1} \notin \text{sp}(\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_i) \). Set \( B_{i+1} = \{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_i, \tilde{w}_{i+1}\} \). Then for

\[
  r_1 \tilde{w}_1 + r_2 \tilde{w}_2 + \cdots + r_i \tilde{w}_i + r_{i+1} \tilde{w}_{i+1} = 0,
\]

if \( r_{i+1} \neq 0 \) then \( \tilde{w}_{i+1} = (-r_i/r_{i+1}) \tilde{w}_i + (-r_2/r_{i+1}) \tilde{w}_2 + \cdots + (-r_1/r_{i+1}) \tilde{w}_1 \), contradicting the choice of \( \tilde{w}_{i+1} \notin \text{sp}(\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_i) \). So \( r_{i+1} = 0 \).

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Let \( \tilde{v} \) and \( \tilde{w} \) be independent column vectors in \( \mathbb{R}^n \) and let \( A \) be an invertible \( n \times n \) matrix where \( n > 1 \). Prove that the vectors \( A\tilde{v} \) and \( A\tilde{w} \) are independent.

Solution. We use Definition 2.1, “Linear Dependence and Independence,” to test the set \( \{A\tilde{v}, A\tilde{w}\} \) for linear independence. Suppose \( r_1 A\tilde{v} + r_2 A\tilde{w} = 0 \) for some \( r_1, r_2 \in \mathbb{R} \). By Theorem 1.3.A, “Properties of Matrix Algebra,” we have

\[
  0 = r_1 A\tilde{v} + r_2 A\tilde{w} = A(r_1 \tilde{v} + r_2 \tilde{w}) \quad \text{by Theorem 1.3.A(7)}
\]

\[
  = A(r_1 \tilde{v} + r_2 \tilde{w}) \quad \text{by Theorem 1.3.A(10)}.
\]

Therefore \( 0 = r_1 \tilde{v} + r_2 \tilde{w} \). Since \( \tilde{v} \) and \( \tilde{w} \) are independent then by Definition 2.1 we must have \( r_1 = r_2 = 0 \). That is, \( r_1 A\tilde{v} + r_2 A\tilde{w} = 0 \) implies \( r_1 = r_2 = 0 \). So, again by Definition 2.1, \( A\tilde{v} \) and \( A\tilde{w} \) are independent.
Page 136 Number 38. Prove that if $W$ is a subspace of $\mathbb{R}^n$ and $\dim(W) = n$ then $W = \mathbb{R}^n$.

Solution. Of course $\dim(\mathbb{R}^n) = n$ since the standard basis for $\mathbb{R}^n$ (see Section 1.1) has $n$ vectors. If $W$ is a subspace of $\mathbb{R}^n$ of dimension $n$ then by Definition 2.2, “Dimension of a Subspace,” the number of elements in a basis $B$ of $W$ is $n$. By Theorem 2.1(2), “Alternative Characterization of a Basis,” $B$ is a linearly independent set. So $B$ is a linearly independent set of $n$ vectors and by Theorem 2.3(2), “Existence and Determination of Bases,” $B$ can be enlarged to become a basis for $\mathbb{R}^n$. However, a basis of $\mathbb{R}^n$ contains $n$ vectors and so no additional vectors can be added to $B$ in the creation of such a basis. So $B$ must already be a basis of $\mathbb{R}^n$ and hence $W = \mathbb{R}^n$. \qed