Chapter 2. Dimension, Rank, and Linear Transformations
Section 2.1. Independence and Dimension—Proofs of Theorems
Theorem 2.1

Theorem 2.1. Alternative Characterization of Basis
Let \( W \) be a subspace of \( \mathbb{R}^n \). A subset \( \{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \} \) of \( W \) is a basis for \( W \) if and only if

(1) \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) and
(2) the vectors \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \) are linearly independent.

Proof. Recall that we defined \( \{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \} \) as a basis for \( W \) if every vector in \( W \) can be expressed as a unique linear combination of \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \) (see Definition 1.17).
Theorem 2.1. Alternative Characterization of Basis

Let $W$ be a subspace of $\mathbb{R}^n$. A subset $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ of $W$ is a basis for $W$ if and only if

1. $W = \text{span}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ and
2. the vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ are linearly independent.

Proof. Recall that we defined $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ as a basis for $W$ if every vector in $W$ can be expressed as a unique linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ (see Definition 1.17).

Let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ be a basis for $W$. Then every vector in $W$ is a (unique) linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ and so these vectors span $W$ and (1) holds.
Theorem 2.1

Theorem 2.1. Alternative Characterization of Basis

Let $W$ be a subspace of $\mathbb{R}^n$. A subset $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ of $W$ is a basis for $W$ if and only if

1. $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$ and
2. the vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ are linearly independent.

Proof. Recall that we defined $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ as a basis for $W$ if every vector in $W$ can be expressed as a unique linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ (see Definition 1.17).

Let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ be a basis for $W$. Then every vector in $W$ is a (unique) linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ and so these vectors span $W$ and (1) holds. To show linear independence, we consider the equation $r_1\vec{w}_1 + r_2\vec{w}_2 + \cdots + r_k\vec{w}_k = \vec{0}$. We know that $r_1 = r_2 = \cdots = r_k = 0$ is one possible choice for the $r_i$, but since $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$ is a basis for $W$ then this is the only choice for the $r_i$ since $\vec{0}$ is a unique linear combination of the $\vec{w}_i$. 
Theorem 2.1. Alternative Characterization of Basis

Let \( W \) be a subspace of \( \mathbb{R}^n \). A subset \( \{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \} \) of \( W \) is a basis for \( W \) if and only if

1. \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \) and
2. the vectors \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \) are linearly independent.

**Proof.** Recall that we defined \( \{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \} \) as a basis for \( W \) if every vector in \( W \) can be expressed as a unique linear combination of \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \) (see Definition 1.17).

Let \( \{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \} \) be a basis for \( W \). Then every vector in \( W \) is a (unique) linear combination of \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \) and so these vectors span \( W \) and (1) holds. To show linear independence, we consider the equation

\[
 r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k = \vec{0}.
\]

We know that \( r_1 = r_2 = \cdots = r_k = 0 \) is one possible choice for the \( r_i \), but since \( \{ \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \} \) is a basis for \( W \) then this is the only choice for the \( r_i \) since \( \vec{0} \) is a unique linear combination of the \( \vec{w}_i \).
Theorem 2.1 (continued)

**Proof (continued).** That is, \( r_1 \vec{w}_1 + r_2 \vec{w}_2 r_k \vec{w}_k = \vec{0} \) implies \( r_1 = r_2 = \cdots = r_k = 0 \). So, by Definition 2.1, “Linear Dependence and Independence,” the \( \vec{w}_i \) are not linearly dependent. That is, \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \) are linearly independent and (2) holds.

Now suppose (1) and (2) hold. Then every vector in \( W \) can be expressed as some linear combination of the \( \vec{w}_i \) since the \( \vec{w}_i \) span \( W \) by (1).
Theorem 2.1 (continued)

**Proof (continued).** That is, \( r_1 \vec{w}_1 + r_2 \vec{w}_2 r_k \vec{w}_k = \vec{0} \) implies \( r_1 = r_2 = \cdots = r_k = 0 \). So, by Definition 2.1, “Linear Dependence and Independence,” the \( \vec{w}_i \) are not linearly dependent. That is, \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \) are linearly independent and (2) holds.

Now suppose (1) and (2) hold. Then every vector in \( W \) can be expressed as some linear combination of the \( \vec{w}_i \) since the \( \vec{w}_i \) span \( W \) by (1). To show uniqueness of the linear combinations, suppose \( \vec{v} \in W \) and 
\[
\vec{v} = r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k = s_1 \vec{w}_1 + s_2 \vec{w}_2 + \cdots + s_k \vec{w}_k.
\]
Then 
\[
(r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k) - (s_1 \vec{w}_1 + s_2 \vec{w}_2 + \cdots + s_k \vec{w}_k) = \vec{0}
\]
and 
\[
(r_1 - s_1) \vec{w}_1 + (r_2 - s_2) \vec{w}_2 + \cdots + (r_k - s_k) \vec{w}_k = \vec{0}.
\]
Theorem 2.1 (continued)

**Proof (continued).** That is, \( r_1 \vec{w}_1 + r_2 \vec{w}_2 r_k \vec{w}_k = \vec{0} \) implies \( r_1 = r_2 = \cdots = r_k = 0 \). So, by Definition 2.1, “Linear Dependence and Independence,” the \( \vec{w}_i \) are not linearly dependent. That is, \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \) are linearly independent and (2) holds.

Now suppose (1) and (2) hold. Then every vector in \( W \) can be expressed as some linear combination of the \( \vec{w}_i \) since the \( \vec{w}_i \) span \( W \) by (1). To show uniqueness of the linear combinations, suppose \( \vec{v} \in W \) and \( \vec{v} = r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k = s_1 \vec{w}_1 + s_2 \vec{w}_2 + \cdots + s_k \vec{w}_k \). Then
\[
(r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k) - (s_1 \vec{w}_1 + s_2 \vec{w}_2 + \cdots + s_k \vec{w}_k) = \vec{0} \text{ and } (r_1 - s_1) \vec{w}_1 + (r_2 - s_2) \vec{w}_2 + \cdots + (r_k - s_k) \vec{w}_k = \vec{0}.
\]
Since the \( \vec{w}_i \) are linearly independent by (2), then \( r_1 - s_1 = r_2 - s_2 = \cdots = r_k - s_k = 0 \) by Note 2.1.A and so \( r_1 = s_1, r_2 = s_2, \ldots, r_k = s_k \). That is, there is a unique linear combination of the \( \vec{w}_i \) which equals \( \vec{v} \).
Theorem 2.1 (continued)

**Proof (continued).** That is, \( r_1 \vec{w}_1 + r_2 \vec{w}_2 r_k \vec{w}_k = \vec{0} \) implies \( r_1 = r_2 = \cdots = r_k = 0 \). So, by Definition 2.1, “Linear Dependence and Independence,” the \( \vec{w}_i \) are not linearly dependent. That is, \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \) are linearly independent and (2) holds.

Now suppose (1) and (2) hold. Then every vector in \( W \) can be expressed as some linear combination of the \( \vec{w}_i \) since the \( \vec{w}_i \) span \( W \) by (1). To show uniqueness of the linear combinations, suppose \( \vec{v} \in W \) and \( \vec{v} = r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k = s_1 \vec{w}_1 + s_2 \vec{w}_2 + \cdots + s_k \vec{w}_k \). Then \( (r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k) - (s_1 \vec{w}_1 + s_2 \vec{w}_2 + \cdots + s_k \vec{w}_k) = \vec{0} \) and \( (r_1 - s_1)\vec{w}_1 + (r_2 - s_2)\vec{w}_2 + \cdots + (r_k - s_k)\vec{w}_k = \vec{0} \). Since the \( \vec{w}_i \) are linearly independent by (2), then \( r_1 - s_1 = r_2 - s_2 = \cdots = r_k - s_k = 0 \) by Note 2.1.A and so \( r_1 = s_1, r_2 = s_2, \ldots, r_k = s_k \). That is, there is a unique linear combination of the \( \vec{w}_i \) which equals \( \vec{v} \). Since \( \vec{v} \) is an arbitrary vector in \( W \), then \( \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\} \) is a basis for \( W \). \( \square \)
Theorem 2.1 (continued)

Proof (continued). That is, \( r_1 \vec{w}_1 + r_2 \vec{w}_2 + r_k \vec{w}_k = \vec{0} \) implies \( r_1 = r_2 = \cdots = r_k = 0 \). So, by Definition 2.1, “Linear Dependence and Independence,” the \( \vec{w}_i \) are not linearly dependent. That is, \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \) are linearly independent and (2) holds.

Now suppose (1) and (2) hold. Then every vector in \( W \) can be expressed as some linear combination of the \( \vec{w}_i \) since the \( \vec{w}_i \) span \( W \) by (1). To show uniqueness of the linear combinations, suppose \( \vec{v} \in W \) and \( \vec{v} = r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k = s_1 \vec{w}_1 + s_2 \vec{w}_2 + \cdots + s_k \vec{w}_k \). Then \((r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_k \vec{w}_k) - (s_1 \vec{w}_1 + s_2 \vec{w}_2 + \cdots + s_k \vec{w}_k) = \vec{0} \) and \((r_1 - s_1) \vec{w}_1 + (r_2 - s_2) \vec{w}_2 + \cdots + (r_k - s_k) \vec{w}_k = \vec{0} \). Since the \( \vec{w}_i \) are linearly independent by (2), then \( r_1 - s_1 = r_2 - s_2 = \cdots = r_k - s_k = 0 \) by Note 2.1.A and so \( r_1 = s_1, r_2 = s_2, \ldots, r_k = s_k \). That is, there is a unique linear combination of the \( \vec{w}_i \) which equals \( \vec{v} \). Since \( \vec{v} \) is an arbitrary vector in \( W \), then \( \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\} \) is a basis for \( W \). \( \square \)
Page 134 Number 8. Use Theorem 2.1.A, “Finding a Basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$,” to find a basis for $W = \text{sp}([-3, 1], [9, -3])$.

Solution. We create matrix $A$ with vectors $[-3, 1]$ and $[9, -3]$ as columns:

$$A = \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix}.$$
Page 134 Number 8. Use Theorem 2.1.A, “Finding a Basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$,” to find a basis for $W = \text{sp}([-3, 1], [9, -3])$.

Solution. We create matrix $A$ with vectors $[-3, 1]$ and $[9, -3]$ as columns:

$$A = \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix}.$$  Now we row reduce $A$:

$$A = \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix} \xrightarrow{R_1 \to R_1 /(-3)} \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} = H.$$
Use Theorem 2.1.A, “Finding a Basis for \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \),” to find a basis for \( W = \text{sp}([-3, 1], [9, -3]) \).

**Solution.** We create matrix \( A \) with vectors \([-3, 1]\) and \([9, -3]\) as columns:

\[
A = \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix}.
\]

Now we row reduce \( A \):

\[
\begin{align*}
A &= \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix} \\
\overset{R_1 \rightarrow R_1/(3)}{\sim} &= \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \\
\overset{R_2 \rightarrow R_2 - R_1}{\sim} &= \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} = H.
\end{align*}
\]

Since \( H \) is in row echelon form and has a pivot only in the first column, then by Theorem 2.1.A, \([[-3, 1]]\) is a basis of \( W \). \(\square\)
Use Theorem 2.1.A, “Finding a Basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots , \vec{w}_k)$,” to find a basis for $W = \text{sp}([-3, 1], [9, -3])$.

**Solution.** We create matrix $A$ with vectors $[-3, 1]$ and $[9, -3]$ as columns:

$$A = \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix}.$$  

Now we row reduce $A$:

$$A = \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1/(-3)} \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_1} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} = H.$$

Since $H$ is in row echelon form and has a pivot only in the first column, then by Theorem 2.1.A, $\{[-3, 1]\}$ is a basis of $W$.  $\Box$
Use Theorem 2.1.A, “Finding a Basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$,” to find a basis for $W = \text{sp}([-2, 3, 1], [3, -1, 2], [1, 2, 3], [-1, 5, 4])$ in $\mathbb{R}^3$.

**Solution.** We create matrix $A$ with the vectors in the spanning set as columns: 

$$A = \begin{bmatrix} -2 & 3 & 1 & -1 \\ 3 & -1 & 2 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$
Use Theorem 2.1.A, “Finding a Basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$,” to find a basis for $W = \text{sp}([−2, 3, 1], [3, −1, 2], [1, 2, 3], [−1, 5, 4])$ in $\mathbb{R}^3$.

Solution. We create matrix $A$ with the vectors in the spanning set as columns:

$$A = \begin{bmatrix}
−2 & 3 & 1 & −1 \\
3 & −1 & 2 & 5 \\
1 & 2 & 3 & 4
\end{bmatrix}.$$ 

Now we row reduce $A$:

$$A = \begin{bmatrix}
−2 & 3 & 1 & −1 \\
3 & −1 & 2 & 5 \\
1 & 2 & 3 & 4
\end{bmatrix} R_1 \leftrightarrow R_3 \begin{bmatrix}
1 & 2 & 3 & 4 \\
3 & −1 & 2 & 5 \\
−2 & 3 & 1 & −1
\end{bmatrix}.$$
Use Theorem 2.1.A, “Finding a Basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$,” to find a basis for $W = \text{sp}([-2, 3, 1], [3, -1, 2], [1, 2, 3], [-1, 5, 4])$ in $\mathbb{R}^3$.

**Solution.** We create matrix $A$ with the vectors in the spanning set as columns: $A = \begin{bmatrix} -2 & 3 & 1 & -1 \\ 3 & -1 & 2 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. Now we row reduce $A$:

\[
A = \begin{bmatrix} -2 & 3 & 1 & -1 \\ 3 & -1 & 2 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix} \overset{R_1 \leftrightarrow R_3}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & -1 & 2 & 5 \\ -2 & 3 & 1 & -1 \end{bmatrix} \overset{R_2 \rightarrow R_2 - 3R_1}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -7 & -7 \\ 0 & 7 & 7 & 7 \end{bmatrix} \overset{R_3 \rightarrow R_3 + R_2}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -7 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = H.
\]
Use Theorem 2.1.A, “Finding a Basis for \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \),” to find a basis for \( W = \text{sp}([-2, 3, 1], [3, -1, 2], [1, 2, 3], [-1, 5, 4]) \) in \( \mathbb{R}^3 \).

Solution. We create matrix \( A \) with the vectors in the spanning set as columns: 

\[
A = \begin{bmatrix}
-2 & 3 & 1 & -1 \\
3 & -1 & 2 & 5 \\
1 & 2 & 3 & 4 \\
\end{bmatrix}
\]

Now we row reduce \( A \):

\[
R_1 \leftrightarrow R_3 \\
R_2 \rightarrow R_2 - 3R_1 \\
R_3 \rightarrow R_3 + 2R_1 \\
\begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & -7 & -7 & -7 \\
0 & 7 & 7 & 7 \\
\end{bmatrix} \\
\begin{bmatrix}
1 & 2 & 3 & 4 \\
3 & -1 & 2 & 5 \\
-2 & 3 & 1 & -1 \\
\end{bmatrix} \\
\begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & -7 & -7 & -7 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} = H.
\]
Use Theorem 2.1.A, “Finding a Basis for $W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$,” to find a basis for $W = \text{sp}([-2, 3, 1], [3, -1, 2], [1, 2, 3], [-1, 5, 4])$ in $\mathbb{R}^3$.

Solution (continued).

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -7 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = H.$$

Since $H$ is in row echelon form and has a pivot in each of the first two columns then, by Theorem 2.1.A, a set consisting of the first two vectors, $\vec{w}_1, \vec{w}_2$ is a basis for $W$; that is, $\{[-2, 3, 1], [3, -1, 2]\}$ is a basis for $W$. 
Page 134 Number 10. Use Theorem 2.1.A, “Finding a Basis for $W = \text{span}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$,” to find a basis for $W = \text{span}([-2, 3, 1], [3, -1, 2], [1, 2, 3], [-1, 5, 4])$ in $\mathbb{R}^3$.

Solution (continued).

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -7 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = H.$$  

Since $H$ is in row echelon form and has a pivot in each of the first two columns then, by Theorem 2.1.A, a set consisting of the first two vectors, $\vec{w}_1, \vec{w}_2$ is a basis for $W$; that is, $\{-2, 3, 1], [3, -1, 2]\}$ is a basis for $W$. Notice that the third vector is a linear combination of these two, $1[-2, 3, 1] + 1[3, -1, 2] = [1, 2, 3]$, and the fourth vector is a linear combination of these two, $2[-2, 3, 1] + 1[3, -1, 2] = [-1, 5, 4]$. □
Page 134 Number 10. Use Theorem 2.1.A, “Finding a Basis for \( \mathcal{W} = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \),” to find a basis for \( \mathcal{W} = \text{sp}([-2, 3, 1], [3, -1, 2], [1, 2, 3], [-1, 5, 4]) \) in \( \mathbb{R}^3 \).

Solution (continued).

\[
A \sim \begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & -7 & -7 & -7 \\
0 & 0 & 0 & 0
\end{bmatrix} = H.
\]

Since \( H \) is in row echelon form and has a pivot in each of the first two columns then, by Theorem 2.1.A, a set consisting of the first two vectors, \( \vec{w}_1, \vec{w}_2 \) is a basis for \( \mathcal{W} \); that is, \( \{ [-2, 3, 1], [3, -1, 2] \} \) is a basis for \( \mathcal{W} \). Notice that the third vector is a linear combination of these two, 
\[
1[-2, 3, 1] + 1[3, -1, 2] = [1, 2, 3],
\]
and the fourth vector is a linear combination of these two, 
\[
2[-2, 3, 1] + 1[3, -1, 2] = [-1, 5, 4].
\]
\( \square \)
Determine whether the set 
\{[1, -3, 2], [2, -5, 3], [4, 0, 1]\} is linearly dependent or independent.

Solution. We use Theorem 2.1.A, “Finding a Basis for \( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \),” and test to see if the set of vectors is a basis for its span. Let \( W = \text{sp}([1, -3, 2], [2, -5, 3], [4, 0, 1]) \).
Determine whether the set
\{ [1, -3, 2], [2, -5, 3], [4, 0, 1] \} is linearly dependent or independent.

Solution. We use Theorem 2.1.A, “Finding a Basis for 
\( W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k) \),” and test to see if the set of vectors is a basis for its span. Let \( W = \text{sp}([1, -3, 2], [2, -5, 3], [4, 0, 1]) \). By Theorem 2.1, a basis for a vector space \( W \) is a linearly independent spanning set. Of course the set of vectors spans its span(!), so it is a basis of its span if and only if the set is a linearly independent set of vectors.
Page 135 Number 22. Determine whether the set 
\([1, -3, 2], [2, -5, 3], [4, 0, 1]\) is linearly dependent or independent.

Solution. We use Theorem 2.1.A, “Finding a Basis for 
\(W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)\),” and test to see if the set of vectors is a basis for 
its span. Let \(W = \text{sp}([1, -3, 2], [2, -5, 3], [4, 0, 1])\). By Theorem 2.1, a 
basis for a vector space \(W\) is a linearly independent spanning set. Of 
course the set of vectors spans its span(!), so it is a basis of its span if and 
only if the set is a linearly independent set of vectors. We create matrix \(A\) 
with the vectors in the set as its columns and row reduce:

\[
A = \begin{bmatrix}
1 & 2 & 4 \\
-3 & -5 & 0 \\
2 & 3 & 1 \\
\end{bmatrix}
\]

\[
R_2 \rightarrow R_2 + 3R_1
\]

\[
\begin{bmatrix}
1 & 2 & 4 \\
0 & 1 & 12 \\
0 & -1 & -7 \\
\end{bmatrix}
\]
Page 135 Number 22. Determine whether the set 
\{[1, -3, 2], [2, -5, 3], [4, 0, 1]\} is linearly dependent or independent.

**Solution.** We use Theorem 2.1.A, “Finding a Basis for 
$W = \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k)$,” and test to see if the set of vectors is a basis for 
its span. Let $W = \text{sp}([1, -3, 2], [2, -5, 3], [4, 0, 1])$. By Theorem 2.1, a 
basis for a vector space $W$ is a linearly independent spanning set. Of 
course the set of vectors spans its span(!), so it is a basis of its span if and 
only if the set is a linearly independent set of vectors. We create matrix $A$ 
with the vectors in the set as its columns and row reduce:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & -5 & 0 \\ 2 & 3 & 1 \end{bmatrix} \overset{R_2 \rightarrow R_2 + 3R_1, R_3 \rightarrow R_3 - 2R_1}{\longrightarrow} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 12 \\ 0 & -1 & -7 \end{bmatrix}$$

...
Page 135 Number 22. Determine whether the set 
\{[1, -3, 2], [2, -5, 3], [4, 0, 1]\} is linearly dependent or independent.

Solution (continued). . .

\[
\begin{bmatrix}
1 & 2 & 4 \\
0 & 1 & 12 \\
0 & -1 & -7 \\
\end{bmatrix}
\rightarrow 
\begin{bmatrix}
1 & 2 & 4 \\
0 & 1 & 12 \\
0 & 0 & 5 \\
\end{bmatrix} = H.
\]

Since \(H\) is in row echelon form and has a pivot in each column then by Theorem 2.1.A the set of all three vectors in \{[1, -3, 2], [2, -5, 3], [4, 0, 1]\} form a basis for \(W\). Therefore the set of vectors is linearly independent. \(\square\)
Page 135 Number 22. Determine whether the set 
\{[1, −3, 2], [2, −5, 3], [4, 0, 1]\} is linearly dependent or independent.

Solution (continued)...

\[
\begin{bmatrix}
1 & 2 & 4 \\
0 & 1 & 12 \\
0 & −1 & −7
\end{bmatrix}
\]

\[
R_3 \rightarrow R_3 + R_2
\]

\[
\begin{bmatrix}
1 & 2 & 4 \\
0 & 1 & 12 \\
0 & 0 & 5
\end{bmatrix} = H.
\]

Since \(H\) is in row echelon form and has a pivot in each column then by Theorem 2.1.A the set of all three vectors in \{[1, −3, 2], [2, −5, 3], [4, 0, 1]\} form a basis for \(W\). Therefore the set of vectors is linearly independent. □
Theorem 2.2. Relative Sizes of Spanning and Independent Sets.

Let $W$ be a subspace of $\mathbb{R}^n$. Let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ be vectors in $W$ that span $W$ and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be vectors in $W$ that are independent. Then $k \geq m$.

Proof. We give a proof by contradiction. ASSUME $k < m$. 
Theorem 2.2. Relative Sizes of Spanning and Independent Sets.
Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ be vectors in $\mathcal{W}$ that span $\mathcal{W}$ and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be vectors in $\mathcal{W}$ that are independent. Then $k \geq m$.

**Proof.** We give a proof by contradiction. ASSUME $k < m$. Since the vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ span $\mathcal{W}$ and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are in $\mathcal{W}$ then there are scalars $a_{ij}$ such that:

$$
\vec{v}_1 = a_{11}\vec{w}_1 + a_{21}\vec{w}_2 + \cdots + a_{k1}\vec{w}_k \\
\vec{v}_2 = a_{12}\vec{w}_1 + a_{22}\vec{w}_2 + \cdots + a_{k2}\vec{w}_k \\
\vdots \\
\vec{v}_m = a_{1m}\vec{w}_1 + a_{2m}\vec{w}_2 + \cdots + a_{km}\vec{w}_k
$$
Theorem 2.2. Relative Sizes of Spanning and Independent Sets.

Let $W$ be a subspace of $\mathbb{R}^n$. Let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ be vectors in $W$ that span $W$ and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be vectors in $W$ that are independent. Then $k \geq m$.

Proof. We give a proof by contradiction. ASSUME $k < m$. Since the vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ span $W$ and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are in $W$ then there are scalars $a_{ij}$ such that:

\[
\begin{align*}
\vec{v}_1 &= a_{11} \vec{w}_1 + a_{21} \vec{w}_2 + \cdots + a_{k1} \vec{w}_k \\
\vec{v}_2 &= a_{12} \vec{w}_1 + a_{22} \vec{w}_2 + \cdots + a_{k2} \vec{w}_k \\
&\vdots \\
\vec{v}_m &= a_{1m} \vec{w}_1 + a_{2m} \vec{w}_2 + \cdots + a_{km} \vec{w}_k
\end{align*}
\]
Theorem 2.2 (continued 1)

Proof (continued). We introduce coefficients \( x_1, x_2, \ldots, x_m \) of \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \) as follows:

\[
\begin{align*}
    x_1 \vec{v}_1 &= a_{11} x_1 \vec{w}_1 + a_{21} x_1 \vec{w}_2 + \cdots + a_{k1} x_1 \vec{w}_k \\
    x_2 \vec{v}_2 &= a_{12} x_2 \vec{w}_1 + a_{22} x_2 \vec{w}_2 + \cdots + a_{k2} x_2 \vec{w}_k \\
    \vdots &\vdots \vdots \\
    x_m \vec{v}_m &= a_{1m} x_m \vec{w}_1 + a_{2m} x_m \vec{w}_2 + \cdots + a_{km} x_m \vec{w}_k
\end{align*}
\]

Now summing these equation we get

\[
\begin{align*}
    x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_m \vec{v}_m &= (a_{11} x_1 + a_{12} x_2 + \cdots + a_{1m} x_m) \vec{w}_1 \\
    &+ (a_{21} x_1 + a_{22} x_2 + \cdots + a_{2m} x_m) \vec{w}_2 + \cdots + (a_{k1} x_1 + a_{k2} x_2 + \cdots + a_{km} x_m) \vec{w}_k
\end{align*}
\]

Consider the system of equations (which results by requiring each coefficient of the \( \vec{w}_i \)'s to be 0): \( \ldots \)
Theorem 2.2 (continued 1)

Proof (continued). We introduce coefficients $x_1, x_2, \ldots, x_m$ of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ as follows:

\[
\begin{align*}
    x_1 \vec{v}_1 &= a_{11}x_1 \vec{w}_1 + a_{21}x_1 \vec{w}_2 + \cdots + a_{k1}x_1 \vec{w}_k \\
    x_2 \vec{v}_2 &= a_{12}x_2 \vec{w}_1 + a_{22}x_2 \vec{w}_2 + \cdots + a_{k2}x_2 \vec{w}_k \\
    &\vdots &
    &\vdots \\
    x_m \vec{v}_m &= a_{1m}x_m \vec{w}_1 + a_{2m}x_m \vec{w}_2 + \cdots + a_{km}x_m \vec{w}_k
\end{align*}
\]

Now summing these equation we get

\[
\begin{align*}
    x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_m \vec{v}_m &= (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m) \vec{w}_1 \\
    &+ (a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m) \vec{w}_2 + \cdots + (a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{km}x_m) \vec{w}_k.
\end{align*}
\]

Consider the system of equations (which results by requiring each coefficient of the $\vec{w}_i$’s to be 0): \ldots
Theorem 2.2 (continued 2)

Proof (continued).

\[ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = 0 \]
\[ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = 0 \]
\[ \vdots \]
\[ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{km}x_m = 0 \]

But this is then a system of \( k \) equations in \( m \) unknowns where \( k < m \). By Corollary 2, “Fewer Equations than Unknowns, The Homogeneous Case,” to Theorem 1.17, this system of equations has a nontrivial solution (that is, there are scalars \( x_1, x_2, \ldots, x_m \) where some \( x_i \) is nonzero satisfying all \( m \) equations).
Proof (continued). ... 

\[ \begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= 0 \\
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Theorem 2.2 (continued 2)

Proof (continued). . .

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= 0 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= 0 \\
  &\vdots \nonumber \\
  a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{km}x_m &= 0 
\end{align*}
\]

But this is then a system of \( k \) equations in \( m \) unknowns where \( k < m \). By Corollary 2, “Fewer Equations than Unknowns, The Homogeneous Case,” to Theorem 1.17, this system of equations has a nontrivial solution (that is, there are scalars \( x_1, x_2, \ldots, x_m \) where some \( x_i \) is nonzero satisfying all \( m \) equations). But then we have \( x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_m \vec{v}_m = \vec{0} \) where some \( x_i \) is nonzero. This implies by Definition 2.1, “Linear Dependence and Independence,” that the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \) are linearly dependent, CONTRADICTING the hypothesis that the \( \vec{v}_i \) are independent. So the assumption that \( k < m \) is false and hence \( k \geq m \), as claimed. \( \square \)
Theorem 2.2 (continued 2)

Proof (continued). . .

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= 0 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= 0 \\
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  a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{km}x_m &= 0
\end{align*}
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But this is then a system of \( k \) equations in \( m \) unknowns where \( k < m \). By Corollary 2, “Fewer Equations than Unknowns, The Homogeneous Case,” to Theorem 1.17, this system of equations has a nontrivial solution (that is, there are scalars \( x_1, x_2, \ldots, x_m \) where some \( x_i \) is nonzero satisfying all \( m \) equations). But then we have \( x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{0} \) where some \( x_i \) is nonzero. This implies by Definition 2.1, “Linear Dependence and Independence,” that the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \) are linearly dependent, CONTRADICTING the hypothesis that the \( \vec{v}_i \) are independent. So the assumption that \( k < m \) is false and hence \( k \geq m \), as claimed. \( \square \)
**Corollary 2.1.A. Invariance of Dimension.**

Any two bases of a subspace of $\mathbb{R}^n$ contains the same number of vectors.

**Proof.** Suppose that both $B$, a set of $k$ vectors, and $B'$, a set of $m$ vectors, are bases for $W$. Then both $B$ and $B'$ are linearly independent spanning sets of $W$ by Theorem 2.1, “Alternative Characterization of a Basis.”
Corollary 2.1.A

**Corollary 2.1.A. Invariance of Dimension.**
Any two bases of a subspace of $\mathbb{R}^n$ contains the same number of vectors.

**Proof.** Suppose that both $B$, a set of $k$ vectors, and $B'$, a set of $m$ vectors, are bases for $W$. Then both $B$ and $B'$ are linearly independent spanning sets of $W$ by Theorem 2.1, “Alternative Characterization of a Basis.” This means that $B$ is a set of $k$ vectors spanning $W$ and $B'$ is a set of $m$ linearly independent vectors in $W$. So by Theorem 2.2, “Relative Sizes of Spanning and Independent Sets,” $k \geq m$. 
Corollary 2.1.A. Invariance of Dimension.

Any two bases of a subspace of $\mathbb{R}^n$ contains the same number of vectors.

**Proof.** Suppose that both $B$, a set of $k$ vectors, and $B'$, a set of $m$ vectors, are bases for $W$. Then both $B$ and $B'$ are linearly independent spanning sets of $W$ by Theorem 2.1, “Alternative Characterization of a Basis.” This means that $B$ is a set of $k$ vectors spanning $W$ and $B'$ is a set of $m$ linearly independent vectors in $W$. So by Theorem 2.2, “Relative Sizes of Spanning and Independent Sets,” $k \geq m$. But also $B'$ is a set of $m$ vectors spanning $W$ and $B$ is a set of $k$ linearly independent vectors in $W$. So by Theorem 2.2, $m \geq k$. Therefore $k = m$ and the bases $B$ and $B'$ have the same number of vectors. Since $B$ and $B'$ are arbitrary bases of $W$, the result follows. $\square$
Corollary 2.1.A. Invariance of Dimension.
Any two bases of a subspace of $\mathbb{R}^n$ contains the same number of vectors.

Proof. Suppose that both $B$, a set of $k$ vectors, and $B'$, a set of $m$ vectors, are bases for $W$. Then both $B$ and $B'$ are linearly independent spanning sets of $W$ by Theorem 2.1, “Alternative Characterization of a Basis.” This means that $B$ is a set of $k$ vectors spanning $W$ and $B'$ is a set of $m$ linearly independent vectors in $W$. So by Theorem 2.2, “Relative Sizes of Spanning and Independent Sets,” $k \geq m$. But also $B'$ is a set of $m$ vectors spanning $W$ and $B$ is a set of $k$ linearly independent vectors in $W$. So by Theorem 2.2, $m \geq k$. Therefore $k = m$ and the bases $B$ and $B'$ have the same number of vectors. Since $B$ and $B'$ are arbitrary bases of $W$, the result follows.
Theorem 2.3(1)

Theorem 2.3. Existence and Determination of Bases.
(1) Every subspace $W \neq \{\vec{0}\}$ of $\mathbb{R}^n$ has a basis and $\dim(W) \leq n$.

Proof. Let $W$ be a subspace of $\mathbb{R}^n$ where $W \neq \{\vec{0}\}$. Then there is some $\vec{w}_1 \in W$ such that $\vec{w}_1 \neq \vec{0}$. Set $B_1 = \{\vec{w}_1\}$. 
Theorem 2.3(1)

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Theorem 2.3(1). Existence and Determination of Bases.

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**Proof.** Let $W$ be a subspace of $\mathbb{R}^n$ where $W \neq \{\vec{0}\}$. Then there is some $\vec{w}_1 \in W$ such that $\vec{w}_1 \neq \vec{0}$. Set $B_1 = \{\vec{w}_1\}$. If $W = \text{sp}(\vec{w}_1)$ then stop, otherwise there is $\vec{w}_2 \in W$ such that $\vec{w}_2 \not\in \text{sp}(\vec{w}_1)$. Set $B_2 = \{\vec{w}_1, \vec{w}_2\}$.

Notice that $\vec{w}_1$ and $\vec{w}_2$ are linearly independent since $r_1\vec{w}_1 + r_2\vec{w}_2 = \vec{0}$ for $r_1 \neq 0$ implies $\vec{w}_1 = (-r_2/r_1)\vec{w}_2$, contradicting the choice of $\vec{w}_2 \not\in \text{sp}(\vec{w}_1)$ (and similarly if $r_2 \neq 0$). If $W = \text{sp}(\vec{w}_1, \vec{w}_2)$ then stop. Otherwise, continue inductively so that if $W \neq \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i)$ where $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i$ are linearly independent, then there is $\vec{w}_{i+1} \in W$ such that $\vec{w}_{i+1} \not\in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i)$. Set $B_{i+1} = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i, \vec{w}_{i+1}\}$. 
Theorem 2.3. Existence and Determination of Bases.

(1) Every subspace $W \neq \{\vec{0}\}$ of $\mathbb{R}^n$ has a basis and $\dim(W) \leq n$.

**Proof.** Let $W$ be a subspace of $\mathbb{R}^n$ where $W \neq \{\vec{0}\}$. Then there is some $\vec{w}_1 \in W$ such that $\vec{w}_1 \neq \vec{0}$. Set $B_1 = \{\vec{w}_1\}$. If $W = \text{sp}(\vec{w}_1)$ then stop, otherwise there is $\vec{w}_2 \in W$ such that $\vec{w}_2 \not\in \text{sp}(\vec{w}_1)$. Set $B_2 = \{\vec{w}_1, \vec{w}_2\}$.

Notice that $\vec{w}_1$ and $\vec{w}_2$ are linearly independent since $r_1 \vec{w}_1 + r_2 \vec{w}_2 = \vec{0}$ for $r_1 \neq 0$ implies $\vec{w}_1 = (-r_2/r_1) \vec{w}_2$, contradicting the choice of $\vec{w}_2 \not\in \text{sp}(\vec{w}_1)$ (and similarly if $r_2 \neq 0$). If $W = \text{sp}(\vec{w}_1, \vec{w}_2)$ then stop. Otherwise, continue inductively so that if $W \neq \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i)$ where $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i$ are linearly independent, then there is $\vec{w}_{i+1} \in W$ such that $\vec{w}_{i+1} \not\in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i)$. Set $B_{i+1} = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i, \vec{w}_{i+1}\}$. Then for

$$r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_i \vec{w}_i + r_{i+1} \vec{w}_{i+1} = \vec{0},$$

if $r_{i+1} \neq 0$ then $\vec{w}_{i+1} = (-r_1/r_{i+1}) \vec{w}_1 + (-r_2/r_{i+1}) \vec{w}_2 + \cdots + (-r_i/r_{i+1}) \vec{w}_i$, contradicting the choice of $\vec{w}_{i+1} \not\in \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i)$. So $r_{i+1} = 0$. 
Theorem 2.3(1). Existence and Determination of Bases.

(1) Every subspace $W \neq \{\vec{0}\}$ of $\mathbb{R}^n$ has a basis and $\dim(W) \leq n$.

Proof. Let $W$ be a subspace of $\mathbb{R}^n$ where $W \neq \{\vec{0}\}$. Then there is some $\vec{w}_1 \in W$ such that $\vec{w}_1 \neq \vec{0}$. Set $B_1 = \{\vec{w}_1\}$. If $W = \text{sp}(\vec{w}_1)$ then stop, otherwise there is $\vec{w}_2 \in W$ such that $\vec{w}_2 \notin \text{sp}(\vec{w}_1)$. Set $B_2 = \{\vec{w}_1, \vec{w}_2\}$. Notice that $\vec{w}_1$ and $\vec{w}_2$ are linearly independent since $r_1 \vec{w}_1 + r_2 \vec{w}_2 = \vec{0}$ for $r_1 \neq 0$ implies $\vec{w}_1 = (-r_2/r_1)\vec{w}_2$, contradicting the choice of $\vec{w}_2 \notin \text{sp}(\vec{w}_1)$ (and similarly if $r_2 \neq 0$). If $W = \text{sp}(\vec{w}_1, \vec{w}_2)$ then stop. Otherwise, continue inductively so that if $W \neq \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i)$ where $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i$ are linearly independent, then there is $\vec{w}_{i+1} \in W$ such that $\vec{w}_{i+1} \notin \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i)$. Set $B_{i+1} = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{i}, \vec{w}_{i+1}\}$. Then for

$$r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_i \vec{w}_i + r_{i+1} \vec{w}_{i+1} = \vec{0},$$

if $r_{i+1} \neq 0$ then $\vec{w}_{i+1} = (-r_1/r_{i+1})\vec{w}_1 + (-r_2/r_{i+1})\vec{w}_2 + \cdots + (-r_i/r_{i+1})\vec{w}_i$, contradicting the choice of $\vec{w}_{i+1} \notin \text{sp}(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i)$. So $r_{i+1} = 0$. 


Theorem 2.3. Existence and Determination of Bases.

(1) Every subspace \( W \neq \{ \vec{0} \} \) of \( \mathbb{R}^n \) has a basis and \( \dim(W) \leq n \).

Proof (continued). But then \( r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_i \vec{w}_i = \vec{0} \) and so \( r_1 = r_2 = \cdots = r_i = 0 \) since \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i \) are linearly independent. So \( r_1 = r_2 = \cdots = r_{i+1} \) and by Definition 2.1, "Linear Dependence and Independence," \( \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i, \vec{w}_{i+1} \) are linearly independent.

Now this process of creating linearly independent sets \( B_i \) consisting of \( i \) vectors must stop at some step \( k \leq n \) by Theorem 2.2, "Relative Sizes of Spanning and Independent Sets." Then \( B_k \) is a linearly independent spanning set for \( W \) and so \( B_k \) is a basis for \( W \) by Theorem 2.1, "Alternative Characterization of Basis."
Theorem 2.3. Existence and Determination of Bases. 

(1) Every subspace $W \neq \{ \vec{0} \}$ of $\mathbb{R}^n$ has a basis and $\dim(W) \leq n$.

Proof (continued). But then $r_1 \vec{w}_1 + r_2 \vec{w}_2 + \cdots + r_i \vec{w}_i = \vec{0}$ and so $r_1 = r_2 = \cdots = r_i = 0$ since $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i$ are linearly independent. So $r_1 = r_2 = \cdots = r_{i+1}$ and by Definition 2.1, “Linear Dependence and Independence,” $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_i, \vec{w}_{i+1}$ are linearly independent.

Now this process of creating linearly independent sets $B_i$ consisting of $i$ vectors must stop at some step $k \leq n$ by Theorem 2.2, “Relative Sizes of Spanning and Independent Sets.” Then $B_k$ is a linearly independent spanning set for $W$ and so $B_k$ is a basis for $W$ by Theorem 2.1, “Alternative Characterization of Basis.”
Page 136 Number 34. Let \( \vec{v} \) and \( \vec{w} \) be independent column vectors in \( \mathbb{R}^n \) and let \( A \) be an invertible \( n \times n \) matrix where \( n > 1 \). Prove that the vectors \( A\vec{v} \) and \( A\vec{w} \) are independent.

Solution. We use Definition 2.1, “Linear Dependence and Independence,” to test the set \( \{A\vec{v}, A\vec{w}\} \) for linear independence. Suppose \( r_1A\vec{v} + r_2A\vec{w} = \vec{0} \) for some \( r_1, r_2 \in \mathbb{R} \).
Page 136 Number 34. Let \( \vec{v} \) and \( \vec{w} \) be independent column vectors in \( \mathbb{R}^n \) and let \( A \) be an invertible \( n \times n \) matrix where \( n > 1 \). Prove that the vectors \( A\vec{v} \) and \( A\vec{w} \) are independent.

**Solution.** We use Definition 2.1, “Linear Dependence and Independence,” to test the set \( \{A\vec{v}, A\vec{w}\} \) for linear independence. Suppose \( r_1 A\vec{v} + r_2 A\vec{w} = \vec{0} \) for some \( r_1, r_2 \in \mathbb{R} \). By Theorem 1.3.A, “Properties of Matrix Algebra,” we have

\[
\begin{align*}
\vec{0} &= r_1 A\vec{v} + r_2 A\vec{w} \\
&= A(r_1 \vec{v}) + A(r_2 \vec{w}) \text{ by Theorem 1.3.A(7)} \\
&= A(r_1 \vec{v} + r_2 \vec{w}) \text{ by Theorem 1.3.A(10)}.
\end{align*}
\]
Page 136 Number 34. Let $\vec{v}$ and $\vec{w}$ be independent column vectors in $\mathbb{R}^n$ and let $A$ be an invertible $n \times n$ matrix where $n > 1$. Prove that the vectors $A\vec{v}$ and $A\vec{w}$ are independent.

Solution. We use Definition 2.1, “Linear Dependence and Independence,” to test the set \{ $A\vec{v}, A\vec{w}$ \} for linear independence. Suppose $r_1A\vec{v} + r_2A\vec{w} = \vec{0}$ for some $r_1, r_2 \in \mathbb{R}$. By Theorem 1.3.A, “Properties of Matrix Algebra,” we have

$$\vec{0} = r_1A\vec{v} + r_2A\vec{w}$$
$$= A(r_1\vec{v}) + A(r_2\vec{w}) \text{ by Theorem 1.3.A(7)}$$
$$= A(r_1\vec{v} + r_2\vec{w}) \text{ by Theorem 1.3.A(10).}$$
Let $\vec{v}$ and $\vec{w}$ be independent column vectors in $\mathbb{R}^n$ and let $A$ be an invertible $n \times n$ matrix where $n > 1$. Prove that the vectors $A\vec{v}$ and $A\vec{w}$ are independent.

Solution (continued). Since $A$ is invertible, then we can multiply both sides of this equation by $A^{-1}$ to get

$$A^{-1}(0) = A^{-1}(A(r_1\vec{v} + r_2\vec{w})) = (A^{-1}A)(r_1\vec{v} + r_2\vec{w})) \text{ by Theorem 1.3.A(8)} = I(r_1\vec{v} + r_2\vec{w}) = r_1\vec{v} + r_2\vec{w}.$$ 

Therefore $\vec{0} = r_1\vec{v} + r_2\vec{w}$. 
Page 136 Number 34. Let $\vec{v}$ and $\vec{w}$ be independent column vectors in $\mathbb{R}^n$ and let $A$ be an invertible $n \times n$ matrix where $n > 1$. Prove that the vectors $A\vec{v}$ and $A\vec{w}$ are independent.

Solution (continued). Since $A$ is invertible, then we can multiply both sides of this equation by $A^{-1}$ to get

$$A^{-1}(\vec{0}) = A^{-1}(A(r_1\vec{v} + r_2\vec{w})) = (A^{-1}A)(r_1\vec{v} + r_2\vec{w}))$$

by Theorem 1.3.A(8)

$$= I(r_1\vec{v} + r_2\vec{w}) = r_1\vec{v} + r_2\vec{w}.$$ 

Therefore $\vec{0} = r_1\vec{v} + r_2\vec{w}$. Since $\vec{v}$ and $\vec{w}$ are independent then by Definition 2.1 we must have $r_1 = r_2 = 0$. That is, $r_1A\vec{v} + r_2A\vec{w} = \vec{0}$ implies $r_1 = r_2 = 0$. So, again by Definition 2.1, $A\vec{v}$ and $A\vec{w}$ are independent.
Let \( \vec{v} \) and \( \vec{w} \) be independent column vectors in \( \mathbb{R}^n \) and let \( A \) be an invertible \( n \times n \) matrix where \( n > 1 \). Prove that the vectors \( A\vec{v} \) and \( A\vec{w} \) are independent.

**Solution (continued).** Since \( A \) is invertible, then we can multiply both sides of this equation by \( A^{-1} \) to get

\[
A^{-1}(\vec{0}) = A^{-1}(A(r_1\vec{v} + r_2\vec{w})) = (A^{-1}A)(r_1\vec{v} + r_2\vec{w})) \text{ by Theorem 1.3.A(8)} = I(r_1\vec{v} + r_2\vec{w}) = r_1\vec{v} + r_2\vec{w}.
\]

Therefore \( \vec{0} = r_1\vec{v} + r_2\vec{w} \). Since \( \vec{v} \) and \( \vec{w} \) are independent then by Definition 2.1 we must have \( r_1 = r_2 = 0 \). That is, \( r_1A\vec{v} + r_2A\vec{w} = \vec{0} \) implies \( r_1 = r_2 = 0 \). So, again by Definition 2.1,\( A\vec{v} \) and \( A\vec{w} \) are independent.
Prove that if \( W \) is a subspace of \( \mathbb{R}^n \) and \( \dim(W) = n \) then \( W = \mathbb{R}^n \).

**Solution.** Of course \( \dim(\mathbb{R}^n) = n \) since the standard basis for \( \mathbb{R}^n \) (see Section 1.1) has \( n \) vectors. If \( W \) is a subspace of \( \mathbb{R}^n \) of dimension \( n \) then by Definition 2.2, “Dimension of a Subspace,” the number of elements in a basis \( B \) of \( W \) is \( n \).
Page 136 Number 38. Prove that if $W$ is a subspace of $\mathbb{R}^n$ and $\dim(W) = n$ then $W = \mathbb{R}^n$.

**Solution.** Of course $\dim(\mathbb{R}^n) = n$ since the standard basis for $\mathbb{R}^n$ (see Section 1.1) has $n$ vectors. If $W$ is a subspace of $\mathbb{R}^n$ of dimension $n$ then by Definition 2.2, “Dimension of a Subspace,” the number of elements in a basis $B$ of $W$ is $n$. By Theorem 2.1(2), “Alternative Characterization of a Basis,” $B$ is a linearly independent set. So $B$ is a linearly independent set of $n$ vectors and by Theorem 2.3(2), “Existence and Determination of Bases,” $B$ can be enlarged to become a basis for $\mathbb{R}^n$. 
Page 136 Number 38. Prove that if $W$ is a subspace of $\mathbb{R}^n$ and $\dim(W) = n$ then $W = \mathbb{R}^n$.

Solution. Of course $\dim(\mathbb{R}^n) = n$ since the standard basis for $\mathbb{R}^n$ (see Section 1.1) has $n$ vectors. If $W$ is a subspace of $\mathbb{R}^n$ of dimension $n$ then by Definition 2.2, “Dimension of a Subspace,” the number of elements in a basis $B$ of $W$ is $n$. By Theorem 2.1(2), “Alternative Characterization of a Basis,” $B$ is a linearly independent set. So $B$ is a linearly independent set of $n$ vectors and by Theorem 2.3(2), “Existence and Determination of Bases,” $B$ can be enlarged to become a basis for $\mathbb{R}^n$. However, a basis of $\mathbb{R}^n$ contains $n$ vectors and so no additional vectors can be added to $B$ in the creation of such a basis. So $B$ must already be a basis of $\mathbb{R}^n$ and hence $W = \mathbb{R}^n$. \qed
**Page 136 Number 38.** Prove that if $W$ is a subspace of $\mathbb{R}^n$ and $\dim(W) = n$ then $W = \mathbb{R}^n$.

**Solution.** Of course $\dim(\mathbb{R}^n) = n$ since the standard basis for $\mathbb{R}^n$ (see Section 1.1) has $n$ vectors. If $W$ is a subspace of $\mathbb{R}^n$ of dimension $n$ then by Definition 2.2, “Dimension of a Subspace,” the number of elements in a basis $B$ of $W$ is $n$. By Theorem 2.1(2), “Alternative Characterization of a Basis,” $B$ is a linearly independent set. So $B$ is a linearly independent set of $n$ vectors and by Theorem 2.3(2), “Existence and Determination of Bases,” $B$ can be enlarged to become a basis for $\mathbb{R}^n$. However, a basis of $\mathbb{R}^n$ contains $n$ vectors and so no additional vectors can be added to $B$ in the creation of such a basis. So $B$ must already be a basis of $\mathbb{R}^n$ and hence $W = \mathbb{R}^n$. □