Chapter 2. Dimension, Rank, and Linear Transformations
Section 2.2. The Rank of a Matrix—Proofs of Theorems
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Let $A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix}$. Find (a) $\text{rank}(A)$, (b) a basis for the row space of $A$, (c) a basis for the column space of $A$, (d) a basis for the nullspace of $A$.

**Solution.** We apply the process of Note 2.2.A and row reduce $A$:

\[
A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -4 & 4 & 1 & 4 \\ 0 & 2 & 3 & 1 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix}
\]
Page 140 Number 6. Let \( A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix} \). Find (a) \( \text{rank}(A) \), (b) a basis for the row space of \( A \), (c) a basis for the column space of \( A \), (d) a basis for the nullspace of \( A \).

Solution. We apply the process of Note 2.2.A and row reduce \( A \):

\[
A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix}
\overset{R_1 \leftrightarrow R_2}{\sim}
\begin{bmatrix} -4 & 4 & 1 & 4 \\ 0 & 2 & 3 & 1 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix}
\overset{R_1 \rightarrow R_1 / (-4)}{\sim}
\begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix}
\overset{R_3 \rightarrow R_3 - 3R_1}{\sim}
\begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 11/4 & 3 \\ -4 & 0 & 1 & 2 \end{bmatrix}
\overset{R_4 \rightarrow R_4 + 4R_1}{\sim}
\begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 11/4 & 3 \\ 0 & -4 & 0 & -2 \end{bmatrix}
\]
Let \( A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix} \). Find (a) \( \text{rank}(A) \), (b) a basis for the row space of \( A \), (c) a basis for the column space of \( A \), (d) a basis for the nullspace of \( A \).

**Solution.** We apply the process of Note 2.2.A and row reduce \( A \):

\[
A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix} \quad \overset{R_1 \leftrightarrow R_2}{\sim} \quad \begin{bmatrix} -4 & 4 & 1 & 4 \\ 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 3 & 3 & 2 & 0 \end{bmatrix} \quad \overset{R_1}{\sim} \quad \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix} \quad \overset{R_3 \rightarrow R_3 - 3R_1}{\sim} \quad \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 6 & 11/4 & 3 \\ 0 & -4 & 0 & -2 \end{bmatrix}
\]
Solution (continued).

\[
\begin{bmatrix}
1 & -1 & -1/4 & -1 \\
0 & 2 & 3 & 1 \\
0 & 6 & 11/4 & 3 \\
0 & -4 & 0 & -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -1 & -1/4 & -1 \\
0 & 2 & 3 & 1 \\
0 & 0 & -25/4 & 0 \\
0 & 0 & 6 & 0
\end{bmatrix}
\]

\[
R_3 \rightarrow R_3 - 3R_2
\]

\[
R_4 \rightarrow R_4 + 2R_2
\]

\[
R_4 \rightarrow R_4 + (24/25)R_3
\]

Since \( H \) is in row echelon form and has pivots in the first three columns we can apply Note 2.2.A to see that:

\( \text{rank}(A) = 3 \)
Solution (continued).

\[
\begin{bmatrix}
1 & -1 & -1/4 & -1 \\
0 & 2 & 3 & 1 \\
0 & 6 & 11/4 & 3 \\
0 & -4 & 0 & -2 \\
\end{bmatrix}
\]

\[
\begin{align*}
R_3 & \rightarrow R_3 - 3 R_2 \\
R_4 & \rightarrow R_4 + 2 R_2 \\
R_4 & \rightarrow R_4 + (24/25) R_3 \\
\end{align*}
\]

\[
\begin{bmatrix}
1 & -1 & -1/4 & -1 \\
0 & 2 & 3 & 1 \\
0 & 0 & -25/4 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

= \( H \).

Since \( H \) is in row echelon form and has pivots in the first three columns we can apply Note 2.2.A to see that:

(a) \( \text{rank}(A) = 3 \) (the number of pivots in \( H \)).
Solution (continued).

\[
\begin{bmatrix}
1 & -1 & -1/4 & -1 \\
0 & 2 & 3 & 1 \\
0 & 6 & 11/4 & 3 \\
0 & -4 & 0 & -2
\end{bmatrix}
\]

- \[R_3 \rightarrow R_3 - 3R_2\]
- \[R_4 \rightarrow R_4 + 2R_2\]
- \[R_4 \rightarrow R_4 + (24/25)R_3\]

\[
\begin{bmatrix}
1 & -1 & -1/4 & -1 \\
0 & 2 & 3 & 1 \\
0 & 0 & -25/4 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} = H.
\]

Since \(H\) is in row echelon form and has pivots in the first three columns we can apply Note 2.2.A to see that:

(a) \[\text{rank}(A) = 3\] (the number of pivots in \(H\)),
Solution (continued).
(b) a basis for the row space of $A$ is the nonzero rows of

$$H = \begin{bmatrix}
1 & -1 & -1/4 & -1 \\
0 & 2 & 3 & 1 \\
0 & 0 & -25/4 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},$$

$$\{[1, -1, -1/4, -1], [0, 2, 3, 1], [0, 0, -25/4, 0]\} \quad \text{(of course we could clean this up by multiplying the first and third vectors by 4 and getting the basis } \{[4, -4, -1, -4], [0, 2, 3, 1], [0, 0, -25, 0]\})$$,
Solution (continued).

(c) a basis for the column space of \( A = \begin{bmatrix} 0 & 2 & 3 & 1 \\ -4 & 4 & 1 & 4 \\ 3 & 3 & 2 & 0 \\ -4 & 0 & 1 & 2 \end{bmatrix} \) is given by the columns of \( A \) corresponding to columns of \( H = \begin{bmatrix} 1 & -1 & -1/4 & -1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -25/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \) which contain pivots, \( \left\{ \begin{bmatrix} 0 \\ -4 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\} \).
Solution (continued). (d) For a basis for the nullspace of $A$, we consider the homogeneous system $A\vec{x} = \vec{0}$, which has (by Theorem 1.6) the same solution as $H\vec{x} = \vec{0}$. To simplify computations, we further row reduce the augmented matrix $[H | \vec{0}]$:

$$[H | \vec{0}] = \begin{bmatrix} 1 & -1 & -1/4 & -1 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & -25/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2/2$$

$$R_3 \rightarrow (-4/25)R_3$$

$$\begin{bmatrix} 1 & -1 & -1/4 & -1 & 0 \\ 0 & 1 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \rightarrow R_1 + R_2$$

$$\ldots$$
Solution (continued). (d) For a basis for the nullspace of $A$, we consider the homogeneous system $A\vec{x} = \vec{0}$, which has (by Theorem 1.6) the same solution as $H\vec{x} = \vec{0}$. To simplify computations, we further row reduce the augmented matrix $[H | \vec{0}]$:

$$
\begin{bmatrix}
1 & -1 & -1/4 & -1 & 0 \\
0 & 2 & 3 & 1 & 0 \\
0 & 0 & -25/4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

After performing the following row operations:

1. $R_2 \rightarrow R_2/2$
2. $R_3 \rightarrow (-4/25)R_3$
3. $R_1 \rightarrow R_1 + R_2$

We obtain:

$$
\begin{bmatrix}
1 & -1 & -1/4 & -1 & 0 \\
0 & 1 & 3/2 & 1/2 & 0 \\
0 & 0 & -1 & 1/10 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Thus, the basis for the nullspace of $A$ is given by the solutions to the system:

$$
\begin{align*}
1 & -1 & -1/4 & -1 & 0 \\
0 & 1 & 3/2 & 1/2 & 0 \\
0 & 0 & -1 & 1/10 & 0 \\
0 & 0 & 0 & 0 & 0
\end{align*}
$$
Solution (continued).

\[
\begin{align*}
R_1 & \rightarrow R_1 + R_2 \\
\text{...} & \\
R_1 & \rightarrow R_1 - (5/4)R_3 \\
R_2 & \rightarrow R_2 - (3/2)R_3
\end{align*}
\]

\[
\begin{bmatrix}
1 & 0 & 5/4 & -1/2 & 0 \\
0 & 1 & 3/2 & 1/2 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & -1/2 & 0 \\
0 & 1 & 0 & 1/2 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\] .

Returning to a system of equations, \ldots
Solution (continued).

\[
\begin{align*}
R_1 & \rightarrow R_1 + R_2 \\
\cdots & \\
R_1 & \rightarrow R_1 - (5/4)R_3 \\
R_2 & \rightarrow R_2 - (3/2)R_3
\end{align*}
\]

\[
\begin{bmatrix}
1 & 0 & 5/4 & -1/2 & 0 \\
0 & 1 & 3/2 & 1/2 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & -1/2 & 0 \\
0 & 1 & 0 & 1/2 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Returning to a system of equations, \ldots
Solution (continued). . .

\[
\begin{align*}
  x_1 &\quad -(1/2)x_4 = 0 &\quad \text{or} &\quad x_1 = (1/2)x_4 \\
  x_2 &\quad +(1/2)x_4 = 0 &\quad &\quad x_2 = -(1/2)x_4 \\
  x_3 &\quad = 0 &\quad &\quad x_3 = 0 \\
  0 &\quad = 0 &\quad &\quad x_4 = x_4.
\end{align*}
\]

With \( r = x_4/2 \) as a free variable we have

\[
\begin{align*}
  x_1 &= (1/2)(2r) = r \\
  x_2 &= -(1/2)(2r) = -r \\
  x_3 &= 0 \\
  x_4 &= 2r
\end{align*}
\]

So the general solution set for the system \( A\vec{x} = \vec{0} \) is

\[
\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} r \in \mathbb{R} \right\}
\]

and so a basis for the nullspace of \( A \) is \( \{[1, -1, 0, 2]^T\} \). \( \square \)
Solution (continued). . .

\[
\begin{align*}
    x_1 - \frac{1}{2}x_4 &= 0 \\
    x_2 + \frac{1}{2}x_4 &= 0 \\
    x_3 &= 0 \\
    x_4 &= 0
\end{align*}
\]

or

\[
\begin{align*}
    x_1 &= \frac{1}{2}x_4 \\
    x_2 &= -\frac{1}{2}x_4 \\
    x_3 &= 0 \\
    x_4 &= x_4.
\end{align*}
\]

With \( r = x_4/2 \) as a free variable we have

\[
\begin{align*}
    x_1 &= \frac{1}{2}(2r) = r \\
    x_2 &= -\frac{1}{2}(2r) = -r \\
    x_3 &= 0 \\
    x_4 &= 2r
\end{align*}
\]

So the general solution set for the system \( A\vec{x} = \vec{0} \) is

\[
\begin{bmatrix}
    r \\
    -1 \\
    0 \\
    2
\end{bmatrix}
\]

and

so a basis for the nullspace of \( A \) is \([1, -1, 0, 2]^T\). □
Page 141 Number 14. Let $A$ and $C$ be matrices such that the product $AC$ is defined. Prove that the column space of $AC$ is contained in the column space of $A$.

Solution. Let $A$ be a $\ell \times m$ matrix, $C$ a $m \times n$ matrix, and let $\vec{v} \in \mathbb{R}^\ell$ be in the column space of $AC$. Then $\vec{v}$ is a linear combination of the columns of $AC$ by the definition of column space (see Section 1.6).
Let $A$ and $C$ be matrices such that the product $AC$ is defined. Prove that the column space of $AC$ is contained in the column space of $A$.

**Solution.** Let $A$ be a $\ell \times m$ matrix, $C$ a $m \times n$ matrix, and let $\vec{v} \in \mathbb{R}^\ell$ be in the column space of $AC$. Then $\vec{v}$ is a linear combination of the columns of $AC$ by the definition of column space (see Section 1.6). So there is some vector $\vec{x} \in \mathbb{R}^m$ such that $(AC)\vec{x} = \vec{v}$, since $(AC)\vec{x}$ is a linear combination of the columns $AC$ with coefficients as the components of $\vec{x}$ (see Note 1.3.A). Now $C\vec{x} \in \mathbb{R}^m$, say $\vec{y} = C\vec{x}$. 


Page 141 Number 14. Let $A$ and $C$ be matrices such that the product $AC$ is defined. Prove that the column space of $AC$ is contained in the column space of $A$.

Solution. Let $A$ be a $\ell \times m$ matrix, $C$ a $m \times n$ matrix, and let $\vec{v} \in \mathbb{R}^\ell$ be in the column space of $AC$. Then $\vec{v}$ is a linear combination of the columns of $AC$ by the definition of column space (see Section 1.6). So there is some vector $\vec{x} \in \mathbb{R}^m$ such that $(AC)\vec{x} = \vec{v}$, since $(AC)\vec{x}$ is a linear combination of the columns $AC$ with coefficients as the components of $\vec{x}$ (see Note 1.3.A). Now $C\vec{x} \in \mathbb{R}^m$, say $\vec{y} = C\vec{x}$. But $\vec{v} = (AC)\vec{x} = A(C\vec{x}) = A\vec{y}$ and $A\vec{y}$ is a linear combination of the columns of $A$ with coefficients as the components of $\vec{y}$. That is, $\vec{v}$ is in the column space of $A$. 
Let $A$ and $C$ be matrices such that the product $AC$ is defined. Prove that the column space of $AC$ is contained in the column space of $A$.

**Solution.** Let $A$ be a $\ell \times m$ matrix, $C$ a $m \times n$ matrix, and let $\vec{v} \in \mathbb{R}^\ell$ be in the column space of $AC$. Then $\vec{v}$ is a linear combination of the columns of $AC$ by the definition of column space (see Section 1.6). So there is some vector $\vec{x} \in \mathbb{R}^m$ such that $(AC)\vec{x} = \vec{v}$, since $(AC)\vec{x}$ is a linear combination of the columns $AC$ with coefficients as the components of $\vec{x}$ (see Note 1.3.A). Now $C\vec{x} \in \mathbb{R}^m$, say $\vec{y} = C\vec{x}$. But $\vec{v} = (AC)\vec{x} = A(C\vec{x}) = A\vec{y}$ and $A\vec{y}$ is a linear combination of the columns of $A$ with coefficients as the components of $\vec{y}$. That is, $\vec{v}$ is in the column space of $A$. So any vector $\vec{v}$ in the column space of $AC$ is in the column space of $A$, and the column space of $A$ contains the column space of $AC$.
Page 141 Number 14. Let $A$ and $C$ be matrices such that the product $AC$ is defined. Prove that the column space of $AC$ is contained in the column space of $A$.

Solution. Let $A$ be a $\ell \times m$ matrix, $C$ a $m \times n$ matrix, and let $\vec{v} \in \mathbb{R}^\ell$ be in the column space of $AC$. Then $\vec{v}$ is a linear combination of the columns of $AC$ by the definition of column space (see Section 1.6). So there is some vector $\vec{x} \in \mathbb{R}^m$ such that $(AC)\vec{x} = \vec{v}$, since $(AC)\vec{x}$ is a linear combination of the columns $AC$ with coefficients as the components of $\vec{x}$ (see Note 1.3.A). Now $C\vec{x} \in \mathbb{R}^m$, say $\vec{y} = C\vec{x}$. But $\vec{v} = (AC)\vec{x} = A(C\vec{x}) = A\vec{y}$ and $A\vec{y}$ is a linear combination of the columns of $A$ with coefficients as the components of $\vec{y}$. That is, $\vec{v}$ is in the column space of $A$. So any vector $\vec{v}$ in the column space of $AC$ is in the column space of $A$, and the column space of $A$ contains the column space of $AC$. \hfill $\square$
Let $A$ and $C$ be matrices such that the product $AC$ is defined. Prove that $\text{rank}(AC) \leq \text{rank}(A)$.

**Solution.** By the definition of rank, $\text{rank}(AC)$ is the dimension of the column space of $AC$ and $\text{rank}(A)$ is the dimension of the column space of $A$. From Exercise 2.2.14 we see that the column space of $AC$ is contained in the column space of $A$. That is, the column space of $AC$ is a subspace of the column space of $A$. Therefore, $\text{rank}(AC) \leq \text{rank}(A)$. 


Let $A$ and $C$ be matrices such that the product $AC$ is defined. Prove that $\text{rank}(AC) \leq \text{rank}(A)$.

**Solution.** By the definition of rank, $\text{rank}(AC)$ is the dimension of the column space of $AC$ and $\text{rank}(A)$ is the dimension of the column space of $A$. From Exercise 2.2.14 we see that the column space of $AC$ is contained in the column space of $A$. That is, the column space of $AC$ is a subspace of the column space of $A$. A basis of the column space of $A$ consists of $\text{rank}(A)$ vectors and by Theorem 2.1(1), “Alternative Characterization of a Basis,” these $\text{rank}(A)$ vectors span the column space of $A$. Now a basis of the column space of $AC$ consists of $\text{rank}(AC)$ vectors and these $\text{rank}(AC)$ vectors are linearly independent by Theorem 2.1(2).
Let $A$ and $C$ be matrices such that the product $AC$ is defined. Prove that $\text{rank}(AC) \leq \text{rank}(A)$.

**Solution.** By the definition of rank, $\text{rank}(AC)$ is the dimension of the column space of $AC$ and $\text{rank}(A)$ is the dimension of the column space of $A$. From Exercise 2.2.14 we see that the column space of $AC$ is contained in the column space of $A$. That is, the column space of $AC$ is a subspace of the column space of $A$. A basis of the column space of $A$ consists of $\text{rank}(A)$ vectors and by Theorem 2.1(1), “Alternative Characterization of a Basis,” these $\text{rank}(A)$ vectors span the column space of $A$. Now a basis of the column space of $AC$ consists of $\text{rank}(AC)$ vectors and these $\text{rank}(AC)$ vectors are linearly independent by Theorem 2.1(2). So the basis of this column space of $AC$ is a set of $\text{rank}(AC)$ linearly independent vectors in the column space of $A$ and so by Theorem 2.2, “Relative Size of Spanning and Independent Sets,” the size of a linearly independent set is less than or equal to the size of a spanning set; hence $\text{rank}(AC) \leq \text{rank}(A)$. $\square$
Page 141 Number 18. Let $A$ and $C$ be matrices such that the product $AC$ is defined. Prove that $\text{rank}(AC) \leq \text{rank}(A)$.

Solution. By the definition of rank, $\text{rank}(AC)$ is the dimension of the column space of $AC$ and $\text{rank}(A)$ is the dimension of the column space of $A$. From Exercise 2.2.14 we see that the column space of $AC$ is contained in the column space of $A$. That is, the column space of $AC$ is a subspace of the column space of $A$. A basis of the column space of $A$ consists of $\text{rank}(A)$ vectors and by Theorem 2.1(1), “Alternative Characterization of a Basis,” these $\text{rank}(A)$ vectors span the column space of $A$. Now a basis of the column space of $AC$ consists of $\text{rank}(AC)$ vectors and these $\text{rank}(AC)$ vectors are linearly independent by Theorem 2.1(2). So the basis of this column space of $AC$ is a set of $\text{rank}(AC)$ linearly independent vectors in the column space of $A$ and so by Theorem 2.2, “Relative Size of Spanning and Independent Sets,” the size of a linearly independent set is less than or equal to the size of a spanning set; hence $\text{rank}(AC) \leq \text{rank}(A)$. □