**Page 144 Example 3**

Let $A$ be an $m \times n$ matrix and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $T_A(\mathbf{x}) = A\mathbf{x}$ for each column vector $\mathbf{x} \in \mathbb{R}^n$. Prove that $T_A$ is a linear transformation.

**Solution.** First, notice that for $m \times n$ matrix $A$ and $n \times 1$ column vector in $\mathbb{R}^n$, we have that $A\mathbf{x}$ is in fact an $m \times 1$ column vector in $\mathbb{R}^m$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and let $r \in \mathbb{R}$ be a scalar. Then we have

$$T_A(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) \text{ by the definition of } T_A$$

$$= A\mathbf{u} + A\mathbf{v} \text{ by Theorem 1.3.A(10), "Distribution Laws"}$$

and

$$T_A(r\mathbf{u}) = A(r\mathbf{u}) \text{ by the definition of } T_A$$

$$= rA\mathbf{u} \text{ by Theorem 1.3.A(7), "Scalars Pull Through"}$$

So $T_A$ satisfies (1) and (2) of Definition 2.3, “Linear Transformation,” and so $T_A$ is a linear transformation.

**Page 152 Number 4**

Is $T([x_1, x_2]) = [x_1 - x_2, x_2 + 1, 3x_1 - 2x_2]$ a linear transformation of $\mathbb{R}^2$ into $\mathbb{R}^3$? Why or why not?

**Solution.** We need to test $T$ to see if it satisfies the definition of “linear transformation.” Let $\mathbf{u} = [u_1, u_2], \mathbf{v} = [v_1, v_2] \in \mathbb{R}^2$. Then

$$T(\mathbf{u} + \mathbf{v}) = T([u_1, u_2] + [v_1, v_2])$$

$$= [(u_1 - u_2) + (v_1 - v_2), (u_2 + 1) + (v_2 + 1), (3u_1 - 2u_2) + (3v_1 - 2v_2)]$$

and

$$T(\mathbf{u}) + T(\mathbf{v}) = T([u_1, u_2]) + T([v_1, v_2])$$

$$= [(u_1 - u_2, u_2 + 1, 3u_1 - 2u_2)] + [(v_1 - v_2, v_2 + 1, 3v_1 - 2v_2)]$$

So $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ if and only if the components of these vectors are equal.
Page 152 Number 4. If \( T([x_1, x_2]) = [x_1 - x_2, x_2 + 1, 3x_1 - 2x_2] \) a linear transformation of \( \mathbb{R}^2 \) into \( \mathbb{R}^3 \)? Why or why not?

Solution (continued). But the second component of \( T(\vec{u}) + T(\vec{v}) = [(u_1 - u_2) + (v_1 - v_2), (u_2 + 1) + (v_2 + 1), (3u_2 - 2u_2) + (3v_1 - 2v_2)] \) is \( u_2 + v_2 + 2 \) and the second component of \( T(\vec{u} + \vec{v}) = [(u_1 - u_2) + (v_1 - v_2), u_2 + v_2 + 1, (3u_1 - 2u_2) + (3v_1 - 2v_2)] \) is \( u_2 + v_2 + 1 \). So the second components are different and \( T(\vec{u} + \vec{v}) \neq T(\vec{u}) + T(\vec{v}) \), so \( T \) fails the definition of linear transformation and \( \boxed{T \text{ is not a linear transformation}} \).

Page 145 Example 4

Page 145 Example 4. Determine all linear transformations of \( \mathbb{R} \) into \( \mathbb{R} \).

Solution. Let \( T : \mathbb{R} \rightarrow \mathbb{R} \) be a linear transformation. Denote \( T([1]) = [m] \), that is, \( T([1]) = [m] \). Then for any \( [x] \in \mathbb{R} \) we have

\[
T([x]) = x[T([1])] = xT([1]) = [mx].
\]

So if \( T : \mathbb{R} \rightarrow \mathbb{R} \) is a linear transformation then \( T([x]) = [mx] \) for some \( m \in \mathbb{R} \).

Theorem 2.7

Theorem 2.7. Bases and Linear Transformations.
Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation and let \( B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\} \) be a basis for \( \mathbb{R}^n \). For any vector \( \vec{v} \in \mathbb{R}^n \), the vector \( T(\vec{v}) \) is uniquely determined by \( T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_n) \).

Proof. Let \( \vec{v} \in \mathbb{R}^n \). Since \( B \) is a basis, then by Definition 2.1, “Linear Dependence and Independence,” there are unique scalars \( r_1, r_2, \ldots, r_n \in \mathbb{R} \) such that \( \vec{v} = r_1 \vec{b}_1 + r_2 \vec{b}_2 + \cdots + r_n \vec{b}_n \). Then by Exercise 32,

\[
T(\vec{v}) = T(r_1 \vec{b}_1 + r_2 \vec{b}_2 + \cdots + r_n \vec{b}_n) = r_1 T(\vec{b}_1) + r_2 T(\vec{b}_2) + \cdots + r_n T(\vec{b}_n).
\]

Since \( r_1, r_2, \ldots, r_n \) are uniquely determined by \( \vec{v} \), then \( T(\vec{v}) \) is completely determined by the vectors \( T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_n) \).

Corollary 2.3.A

Corollary 2.3.A. Standard Matrix Representation of Linear Transformations.
Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be linear, and let \( A \) be the \( m \times n \) matrix whose \( j \)th column is \( T(\hat{e}_j) \). Then \( T(\vec{x}) = A\vec{x} \) for each \( \vec{x} \in \mathbb{R}^n \). \( A \) is the standard matrix representation of \( T \).

Proof. Recall that with \( \hat{e}_j \) as the \( j \)th standard basis vector of \( \mathbb{R}^n \), we have \( A\hat{e}_j \) is the \( j \)th column of \( A \) (see Note 1.3.A) and so \( A\hat{e}_j = T(\hat{e}_j) \). If we define \( T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m \) as \( T_A(\vec{x}) = A\vec{x} \) for all \( \vec{x} \in \mathbb{R}^n \) then \( T_A \) is a linear transformation by Example 3 and \( T \) and \( T_A \) are the same on the standard basis \( \{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\} \) of \( \mathbb{R}^n \). So by Theorem 2.7, “Bases and Linear Transformations,” \( T \) and \( T_A \) are the same linear transformations mapping \( \mathbb{R}^n \rightarrow \mathbb{R}^m \). That is, \( T(\vec{x}) = T_A(\vec{x}) = A\vec{x} \) for all \( \vec{x} \in \mathbb{R}^n \), as claimed.
Theorem 2.3.A. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix representation $A$.

1. The range $T[\mathbb{R}^n]$ of $T$ is the column space of $A$.
2. If $W$ is a subspace of $\mathbb{R}^n$, then $T[W]$ is a subspace of $\mathbb{R}^m$ (i.e. $T$ preserves subspaces).

Proof. (1) Recall that $T[\mathbb{R}^n] = \{ T(x) \mid x \in \mathbb{R}^n \}$. Since $A$ is the standard matrix representation of $T$ then $T[\mathbb{R}^n] = \{ A\vec{x} \mid \vec{x} \in \mathbb{R}^n \}$. Now for $\vec{x} \in \mathbb{R}^n$, $A\vec{x}$ is a linear combination of the columns of $A$ with the components of $\vec{x}$ as the coefficients (see Note 1.3.A) and conversely any linear combination of the columns of $A$ equals $A\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$ (namely, $\vec{x}$ with components equal to the coefficients in the linear combination). So the range of $I$, $I[\mathbb{R}^n]$, consists of precisely the same vectors as the column space of $A$.

(continued)

Theorem 2.3.A. Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix representation $A$.

1. The range $T[\mathbb{R}^n]$ of $T$ is the column space of $A$.
2. If $W$ is a subspace of $\mathbb{R}^n$, then $T[W]$ is a subspace of $\mathbb{R}^m$ (i.e. $T$ preserves subspaces).

Proof (continued). (2) Let $W$ be a subspace of $\mathbb{R}^n$. Then $W$ has a basis by Theorem 2.3(1), “Existence and Determination of Bases,” say $B = \{ b_1, b_2, \ldots, b_k \}$. Now by Exercise 32,

\[ T(r_1 \vec{b}_1 + r_2 \vec{b}_2 + \cdots + r_k \vec{b}_k) = r_1 T(\vec{b}_1) + r_2 T(\vec{b}_2) + \cdots + r_k T(\vec{b}_k) \]

for any $r_1, r_2, \ldots, r_k \in \mathbb{R}$. So $T[W] = \text{span}(T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_k))$ and since the span of a set of vectors in $\mathbb{R}^m$ is a subspace of $\mathbb{R}^m$ by Theorem 1.14, “Subspace Property of a Span,” we have that $T[W]$ is a subspace of $\mathbb{R}^m$. □
Theorem 2.3.B. Matrix Multiplication and Composite Transformations.

A composition of two linear transformations $T$ and $T'$ with standard matrix representation $A$ and $A'$ yields a linear transformation $T' \circ T$ with standard matrix representation $A'A$.

Proof. We have that $T(\bar{x}) = A\bar{x}$ and $T'(\bar{y}) = A'\bar{y}$ for all appropriate $\bar{x}$ and $\bar{y}$ (that is, $\bar{x}$ is the domain of $T$ and $\bar{y}$ in the domain of $T'$). Then for any $\bar{x}$ in the domain of $T$ we have

$$(T' \circ T)(\bar{w}) = T'(T(\bar{x}))$$

by the definition of composition

$$= T'(A\bar{x})$$

since $T(\bar{x}) = A\bar{x}$

$$= A'(A\bar{x})$$

since $T'(\bar{y}) = A'\bar{y}$

$$= (A'A)\bar{x}$$

by Theorem 1.3.A(8),

"Associativity of Matrix Multiplication".

So the standard matrix representation of $T' \circ T$ is $A'A$, as claimed. \(\square\)

Page 153 Number 20

Page 153 Number 20. If $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is defined as $T([x_1, x_2]) = [2x_1 + x_2, x_1, x_1 - x_2]$ and $T' : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is defined by $T'([x_1, x_2, x_3]) = [x_1 - x_2, x_3, x_1 - x_2]$. Find the standard matrix representation for the linear transformation $T \circ T'$ that carries $\mathbb{R}^3$ into $\mathbb{R}^3$. Find a formula for $(T \circ T')([x_1, x_2, x_3])$.

Solution. First, we find the standard matrix representation of $T$ and $T'$.

We have $T([1, 0]) = [2, 1, 1]$ and $T([0, 1]) = [1, 0, -1]$, so by Corollary 2.3.A the standard matrix representation of $T$ is $A_T = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$. Also, $T'([1, 0, 0]) = [1, 1]$, $T'([0, 1, 0]) = [-1, 1]$, and $T'([0, 0, 1]) = [1, 0]$ so the standard matrix representation of $T'$ is $A_{T'} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Page 153 Number 20 (continued)

Solution (continued). By Theorem 2.3.B, the standard matrix representation of $T \circ T'$ is

$$A_TA_{T'} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & -1 & 1 \\ 0 & -2 & 1 \end{bmatrix}.$$ 

By Corollary 2.3.A, a formula for $T \circ T'([x_1, x_2, x_3])$ can be found from

$$A_TA_{T'}\bar{x} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & -1 & 1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - x_2 + 2x_3 \\ x_1 - x_2 + x_3 \\ -2x_2 + x_3 \end{bmatrix},$$

and so $T \circ T'([x_1, x_2, x_3]) = [3x_1 - x_2 + 2x_3, x_1 - x_2 + x_3, -2x_2 + x_3].$ \(\square\)

Page 153 Number 23

Page 153 Number 23. Consider the linear transformation $T([x_1, x_2, x_3]) = [x_1 + x_1 + x_3, x_1 + x_2, x_1]$. Find the standard matrix representation for $T$ and determine if $T$ is invertible. If it is, find a formula for $T^{-1}$ in row notation.

Solution. We find the standard matrix representation for $T$ using Corollary 2.3.A. We have $T([1, 0, 0]) = [1, 1, 1]$, $T([0, 1, 0]) = [1, 1, 0]$, and $T([0, 0, 1]) = [1, 0, 0]$. So the standard matrix representation for $T$ is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

We test $A$ for invertibility:

$$[A \mid I] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

We find $A^{-1}$ and substitute it into the formula $A_{T^{-1}}\bar{x} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 + x_3 \\ x_1 - x_2 - x_3 \\ -x_1 + x_2 + x_3 \end{bmatrix}$ for $T^{-1}([x_1, x_2, x_3]) = \begin{bmatrix} -x_1 + x_2 + x_3 \\ x_1 - x_2 - x_3 \\ -x_1 + x_2 + x_3 \end{bmatrix}$. \(\square\)
Solution (continued).

\[
\begin{align*}
R_3 & \rightarrow R_3 \\
& \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\
& 0 & -1 & -1 & -1 & 0 \\
& 0 & 0 & -1 & -1 & 1 \\
\end{bmatrix} \\
R_2 & \rightarrow R_2 \\
& \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\
& 0 & 1 & 1 & 1 & 0 \\
& 0 & 0 & 1 & 1 & -1 \\
\end{bmatrix} \\
R_1 & \rightarrow R_1 \\
& \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\
& 0 & 1 & 1 & 1 & 0 \\
& 0 & 0 & 1 & 1 & -1 \\
\end{bmatrix} \\
R_2 & \rightarrow R_2 \\
& \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\
& 0 & 1 & 0 & 0 & 1 \\
& 0 & 0 & 1 & 1 & -1 \\
\end{bmatrix}.
\end{align*}
\]

So \( A \) is invertible and \( A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\
& 0 & 1 & -1 \\
& 1 & -1 & 0 \\
\end{bmatrix} \), and so \( T \) is invertible by Theorem 2.3.C.

Page 153 Number 23. Consider the linear transformation
\( T([x_1, x_2, x_3]) = [x_1 + x_1 + x_3, x_1 + x_2, x_1] \). Find the standard matrix representation for \( T \) and determine if \( T \) is invertible. If it is, find a formula for \( T^{-1} \) in row notation.

Solution (continued). Also by Theorem 2.3.C, \( A^{-1} \) is the standard matrix representation of \( T^{-1} \) and we can find the formula for \( T^{-1} \) from
\[
A^{-1}x = \begin{bmatrix} 0 & 0 & 1 \\
& 0 & 1 & -1 \\
& 1 & -1 & 0 \\
\end{bmatrix} \begin{bmatrix} x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} = \begin{bmatrix} x_3 \\
& x_2 - x_3 \\
& x_1 - x_2 \\
\end{bmatrix}.
\]
So \( T^{-1}([x_1, x_2, x_3]) = [x_3, x_2 - x_3, x_1 - x_2] \).