Chapter 2. Dimension, Rank, and Linear Transformations
Section 2.4. Linear Transformations of the Plane—Proofs of Theorems

Solution. Since \( A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \) represents a rotation of \( \mathbb{R}^2 \) about the origin through an angle of \( \theta \), then \( A^3 \) represents a rotation of \( \mathbb{R}^2 \) about the origin through an angle \( 3\theta \). So

\[
\begin{bmatrix} \cos 3\theta & -\sin 3\theta \\ \sin 3\theta & \cos 3\theta \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^3
= \begin{bmatrix} \cos^3 \theta - \sin^2 \theta \cos \theta & -2 \cos \theta \sin \theta \\ 2 \cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}
= \begin{bmatrix} \cos^3 \theta - \cos \theta \sin^2 \theta - 2 \cos \theta \sin^2 \theta & -2 \cos^2 \theta \sin \theta - 2 \cos \theta \sin \theta \cos \theta \\ 2 \cos^2 \theta \sin \theta + \cos \theta \sin^2 \theta & \cos \theta \sin \theta - \sin^3 \theta - 2 \cos \theta \sin \theta \cos \theta \end{bmatrix}.
\]

Hence \( \cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta \) and \( \sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta \).
Solution (continued).

\[
\begin{bmatrix}
1 & -m & 0 \\
m & 1 & 1 \\
\end{bmatrix}
\begin{pmatrix}
R_3-R_2-mR_1 \\
R_3-R_2/(1+m^2) \\
\end{pmatrix}
\begin{bmatrix}
1 & -m & 0 \\
0 & 1 & 1 \\
\end{bmatrix}
\begin{pmatrix}
R_2 & R_3 \\
R_1+R_2 & m/(1+m^2) \\
\end{pmatrix},
\]

so \(c_1 = m/(1+m^2)\) and \(c_2 = 1/(1+m^2)\). Therefore,

\[
T(\vec{e}_1) = T\left(\frac{1}{1+m^2}\vec{b}_1 - \frac{m}{1+m^2}\vec{b}_2\right) = \frac{1}{1+m^2}T(\vec{b}_1) - \frac{m}{1+m^2}T(\vec{b}_2)
\]

\[
= \frac{1}{1+m^2}\vec{b}_1 - \frac{m}{1+m^2}(-\vec{b}_2) = \frac{1}{1+m^2}[1,m] + \frac{m}{1+m^2}[-m,1],
\]

\[
= \begin{bmatrix}
1-m^2 & 2m \\
1+m^2 & 1+m^2
\end{bmatrix}, \ldots
\]

\[
T(\vec{b}_2) = T\left(\frac{m}{1+m^2}\vec{b}_1 + \frac{1}{1+m^2}\vec{b}_2\right) = \frac{m}{1+m^2}T(\vec{b}_1) + \frac{1}{1+m^2}T(\vec{b}_2)
\]

\[
= \frac{m}{1+m^2}\vec{b}_1 + \frac{1}{1+m^2}\vec{b}_2 = \frac{m}{1+m^2}[1,m] - \frac{1}{1+m^2}[-m,1],
\]

\[
= \begin{bmatrix}
2m & m^2-1 \\
1+m^2 & 1+m^2
\end{bmatrix}.
\]

So the matrix \(A\) representing \(T\) is

\[
A = \begin{bmatrix}
\frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\
\frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2}
\end{bmatrix}.
\]

\(\square\)

Page 165 Number 8 (iii, iv)

Page 165 Number 8 (iii, iv). Let \(T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\).

(iii) Show that \(T\) is a vertical expansion followed by a reflection about the x-axis if \(r < -1\).

(iv) Show that \(T\) is a vertical contraction followed by a reflection about the x-axis if \(-1 < r < 0\).

Solution. (iii) If \(r < -1\) then \(|r| > 1\) and so \(A_1 = \begin{bmatrix} 1 & 0 \\ 0 & |r| \end{bmatrix}\) is the standard matrix representation of a linear transformation \(T_1\) which is a vertical expansion. Next, \(X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\) is the standard matrix representation of a linear transformation \(T_1\) which is a reflection about the x-axis. Now

\[
XA_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & |r| \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -|r| \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}
\]

and so \(T\) is a vertical expansion followed by a reflection about the x-axis.

Page 165 Number 8 (iii, iv) (continued)

Page 165 Number 8 (iii, iv). Let \(T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\).

(iii) Show that \(T\) is a vertical expansion followed by a reflection about the x-axis if \(-1 < r < 0\).

Solution (continued). (iv) If \(-1 < r < 0\) then \(0 < |r| < 1\) and so \(A_2 = \begin{bmatrix} 1 & 0 \\ 0 & |r| \end{bmatrix}\) is the standard matrix representation of a linear transformation \(T_2\), which is a vertical contraction. With \(X\) as in part (iii), we have

\[
XA_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & |r| \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -|r| \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}
\]

and so \(T\) is a vertical contraction followed by a reflection about the x-axis. \(\square\)
Theorem 2.4.A

Theorem 2.4.A. Geometric Description of Invertible Transformations of \( \mathbb{R}^2 \).

A linear transformation \( T \) of the plane \( \mathbb{R}^2 \) into itself is invertible if and only if \( T \) consists of a finite sequence of:

- Reflections in the x-axis, the y-axis, or the line \( y = x \);
- Vertical or horizontal expansions or contractions; and
- Vertical or horizontal shears.

Proof. The three elementary row operations correspond to \( 2 \times 2 \) matrices as follows:

1. Row Interchange: \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).
2. Row Scaling: \( B_1 = \begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix} \) and \( B_2 = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix} \).
3. Row Addition: \( C_1 = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \) and \( C_2 = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} \).

Proof (continued). Now \( A \) corresponds to reflection about the line \( y = x \), \( B_1 \) with \( r = -1 \) corresponds to reflection about the y-axis, \( B_1 \) corresponds to a horizontal expansion if \( r > 1 \), \( B_1 \) corresponds to a horizontal contraction if \( 0 < r < 1 \), \( B_1 \) corresponds to a horizontal expansion followed by a reflection about the y-axis if \( r < -1 \) (similar to Exercise 8(iii)), \( B_2 \) corresponds to a horizontal contraction followed by a reflection about the y-axis if \( -1 < r < 0 \) (similar to Exercise 8(iv)), \( B_2 \) with \( r = -1 \) corresponds to reflection about the x-axis, \( B_2 \) corresponds to a vertical expansion if \( r > 1 \), \( B_2 \) corresponds to a vertical contraction if \( 0 < r < 1 \), \( B_2 \) corresponds to a vertical expansion followed by a reflection about the x-axis if \( -1 < r < 0 \) (as shown in Exercise 8(iv)), \( C_1 \) corresponds to a vertical shear, and \( C_2 \) corresponds to a horizontal shear.

Page 165 Number 14

Page 165 Number 14. Consider \( T([x, y]) = [x + y, 2x - y] \). Find the standard matrix representation and write it as a product of elementary matrices. Then describe \( T \) as a sequence of reflections, expansions, contractions, and shears.

Solution. First, \( T([1, 0]) = [1, 2] \) and \( T([0, 1]) = [1, -1] \), so the standard matrix representation of \( T \) is \( A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \). We use the technique of Section 1.5 to write \( A \) as a product of elementary matrices. We have

\[
A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = E_1^{-1},
\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E_2^{-1},
\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E_3^{-1},
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = E_3^{-1}
\]
Page 165 Number 14. Consider \( T([x, y]) = [x + y, 2x - y] \). Find the standard matrix representation and write it as a product of elementary matrices. Then describe \( T \) as a sequence of reflections, expansions, contractions, and shears.

Solution (continued). So

\[
A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.
\]

So \( T \) consist of in order (reading from right to left) a horizontal shear, a vertical expansion and a reflection about the x-axis (see Exercise 8), and a vertical shear. 

Page 166 Number 18. Use algebraic properties of the dot product to compute \( \| \bar{u} - \bar{v} \|^2 = (\bar{u} - \bar{v}) \cdot (\bar{u} - \bar{v}) \), and prove from the resulting equation that a linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) that preserves length also preserves the dot product.

Solution. Let \( \bar{u} \) and \( \bar{v} \) be any vectors in \( \mathbb{R}^2 \). Then

\[
\| \bar{u} - \bar{v} \|^2 = (\bar{u} - \bar{v}) \cdot (\bar{u} - \bar{v}) = \bar{u} \cdot \bar{u} - \bar{v} \cdot \bar{v} = \|\bar{u}\|^2 - 2\bar{u} \cdot \bar{v} + \|\bar{v}\|^2.
\]

Solving for \( \bar{u} \cdot \bar{v} \) gives

\[
\bar{u} \cdot \bar{v} = \frac{1}{2}(\|\bar{u}\|^2 - \|\bar{v}\|^2 - \|\bar{u} - \bar{v}\|^2)
\]

Similarly, \( T(\bar{u}) \cdot T(\bar{v}) = \frac{1}{2}(\| T(\bar{u}) \|^2 + \| T(\bar{v}) \|^2 - \| T(\bar{u} - \bar{v}) \|^2) \).

Page 167 Number 19. Suppose that \( T_A : \mathbb{R}^2 \to \mathbb{R}^2 \) preserves both length and angle. Prove that the two column vectors of the matrix \( A \) are orthogonal unit vectors.

Proof. Since \( A \) is the standard matrix representation of \( T \), the columns of \( A \) are \( T(\hat{e}_1) = T([1, 0]) \) and \( T(\hat{e}_2) = T([0, 1]) \) by Corollary 2.3A, “Standard Matrix Representation of Linear Transformations.” Since \( T_A \) preserves lengths then \( \| T(\hat{e}_1) \| = \| \hat{e}_1 \| = 1 \) and \( \| T(\hat{e}_2) \| = \| \hat{e}_2 \| = 1 \), so the columns of \( A \) are unit vectors. Since \( T \) preserves angles \( \hat{e}_1 \perp \hat{e}_2 \) then \( T(\hat{e}_1) \perp T(\hat{e}_2) \); that is, the columns of \( A \) are orthogonal, as claimed.