Example 3.1.2

Solution. Let \( p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \),
\( q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 \), and
\( r(x) = c_\ell x^\ell + c_{\ell-1} x^{\ell-1} + \cdots + c_1 x + c_0 \) be polynomials in \( \mathcal{P} \) (where, say, \( \ell \leq n \leq m \)) and let \( s \) and \( t \) be real scalars. Then
\[
p(x) + q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_{\ell+1} x^{\ell+1} + (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \cdots + (a_1 + b_1) x + (a_0 + b_0)
\]
(3.1.1)

(As above)
\[
p(x) + q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_{n+1} x^{n+1} + (b_n + a_n) x^n + (b_{n-1} + a_{n-1}) x^{n-1} + \cdots + (b_1 + a_1) x + (b_0 + a_0)
\]
since addition is commutative in \( \mathbb{R} \)
\[
= q(x) + p(x)
\]

Example 3.1.2 (A2)

Solution (continued).

A2.
\[
p(x) + q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_{n+1} x^{n+1} + (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \cdots + (a_1 + b_1) x + (a_0 + b_0)
\]
(3.1.1)

(As above)
\[
p(x) + q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_{n+1} x^{n+1} + (b_n + a_n) x^n + (b_{n-1} + a_{n-1}) x^{n-1} + \cdots + (b_1 + a_1) x + (b_0 + a_0)
\]
since addition is commutative in \( \mathbb{R} \)
\[
= q(x) + p(x)
\]
Example 3.1.2 (A1) (continued)

Solution. A1. (continued) $(p(x) + q(x)) + r(x)$

\[ = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_n x^n + (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \cdots + (a_{\ell+1} + b_{\ell+1}) x^{\ell+1} + \cdots + (a_1 + (b_1 + c_1)) x + (a_0 + (b_0 + c_0)) \]

since addition in $\mathbb{R}$ is associative

\[ = (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 + a_0) + (b_m x^m + b_{m-1} x^{m-1} + \cdots + b_{\ell+1} x^{\ell+1} + (b_{\ell+1} + c_\ell) x^\ell + (b_{\ell-1} + c_{\ell-1}) x^{\ell-1} + \cdots + (b_1 + c_1) x + (b_0 + c_0)) \]

\[ = p(x) + r(x) + (q(x) + r(x)). \]

Notice that, by A2, we can permute $p(x), q(x),$ and $r(x)$ and the associativity claim then holds in general (this is necessary to cover all cases of the relative degrees of the polynomials).

Example 3.1.2 (S1)

Solution (continued).

S1. We have:

\[ s(p(x) + q(x)) \]

\[ = s(b_m x^m + b_{m-1} x^{m-1} + \cdots + b_{n+1} x^{n+1} + (b_n + a_n) x^n + \cdots + b_{\ell+1} x^{\ell+1} + (b_{\ell+1} + c_\ell) x^\ell + (b_{\ell-1} + c_{\ell-1}) x^{\ell-1} + \cdots + (b_1 + c_1) x + (b_0 + c_0)) \]

\[ = s(b_m x^m) + s(b_{m-1} x^{m-1}) + \cdots + s(b_{n+1} x^{n+1}) + s(b_n + a_n) x^n + \cdots + s(b_{\ell+1} x^{\ell+1}) + (b_{\ell+1} + c_\ell) x^\ell + (b_{\ell-1} + c_{\ell-1}) x^{\ell-1} + \cdots + (b_1 + c_1) x + s(b_0 + c_0)) \]

\[ = (sb_m) x^m + (sb_{m-1}) x^{m-1} + \cdots + (sb_{n+1}) x^{n+1} + (sb_n + sa_n) x^n + (sb_{n-1} + sa_{n-1}) x^{n-1} + \cdots + (sb_1 + sa_1) x + (sb_0 + sa_0) \]

since multiplication distributes over addition in $\mathbb{R}$

\[ = sp(x) + sq(x). \]

Example 3.1.2 (A3, A4)

Solution (continued).

A3. We take the zero vector as the polynomial with all coefficients 0: $0(x) = 0$. Then

\[ 0(x) + p(x) = (0 + a_n) x^n + (0 + a_{n-1}) x^{n-1} + \cdots + (0 + a_1) x + (0 + a_0) \]

\[ = a_n x^n - a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \]

since 0 is the additive identity in $\mathbb{R}$

\[ = p(x). \]

A4. For $p(x)$ as given, we define

\[ -p(x) = (-a_n) x^n + (-a_{n-1}) x^{n-1} + \cdots + (-a_1) x + (-a_0). \]

Then

\[ p(x) + (-p(x)) = (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) \]

\[ + (-a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) \]

\[ = (a_n - a_n) x^n + (a_{n-1} - a_{n-1}) x^{n-1} + \cdots + (a_1 - a_1) x + (a_0 - a_0) \]

\[ = 0(x). \]

Example 3.1.2 (S2)

Solution (continued).

S2. We have:

\[ (s + t)p(x) \]

\[ = (s + t)(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) \]

\[ = (s + t)a_n x^n + (s + t)a_{n-1} x^{n-1} + \cdots + (s + t)a_1 x + (s + t)a_0 \]

\[ = (sa_n + ta_n) x^n + (sa_{n-1} + ta_{n-1}) x^{n-1} + \cdots + (sa_1 + ta_1) x + (sa_0 + ta_0) \]

since multiplication distributes over addition in $\mathbb{R}$

\[ = sp(x) + tp(x). \]
**Example 3.1.2 (S3)**

Solution (continued).

**S3.** We have:

\[
\begin{align*}
    s(tp(x)) &= s(t(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)) \\
    &= s(ta_n x^n + ta_{n-1} x^{n-1} + \cdots + ta_1 x + ta_0) \\
    &= s(ta_n)x^n \cdot s(ta_{n-1})x^{n-1} \cdot \cdots \cdot s(ta_1)x \cdot s(ta_0) \\
    &= (st)a_n x^n + (st)a_{n-1} x^{n-1} + \cdots + (st)a_1 x + (st)a_0 \\
    &\quad \text{since multiplication is associative in } \mathbb{R} \\
    &= (st)p(x).
\end{align*}
\]

**Example 3.1.2 (S4)**

Solution (continued).

**S4.** We have:

\[
\begin{align*}
    1p(x) &= 1(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) \\
    &= (1a_n)x^n + (1a_{n-1})x^{n-1} + \cdots + (1a_1)x + (1a_0) \\
    &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\
    &\quad \text{since 1 is the multiplicative identity in } \mathbb{R} \\
    &= p(x).
\end{align*}
\]

So all properties of Definition 3.1 are satisfied and \( \mathcal{P} \) is a vector space. \( \square \)

---

**Page 189 Number 6**

Consider the set \( \mathcal{F} \) of all functions mapping \( \mathbb{R} \) into \( \mathbb{R} \), with scalar multiplication defined for scalar \( r \in \mathbb{R} \) and \( f \in \mathcal{F} \) as

\[
(rf)(x) = rf(x),
\]

and vector addition \( \odot \) defined as

\[
(f \odot g)(x) = \max\{f(x), g(x)\}.
\]

Is \( \mathcal{F} \) a vector space?

**Solution.** The peculiar way of adding vectors yields some problems. For example there can be no additive identity and A3 does not hold. To see this, let \( k \in \mathbb{R} \) be a constant and consider the constant function \( f(x) = k \).

If \( e(x) \) is the additive identity in \( \mathcal{F} \) then

\[
e(x) \odot f(x) = \max\{e(x), f(x)\} = f(x) = k
\]

for all \( x \in \mathbb{R} \). So it must be that \( e(x) \leq k \) for all \( x \in \mathbb{R} \). Since \( k \in \mathbb{R} \) is arbitrary, we have that \( e(x) \leq k \) for all \( k \in \mathbb{R} \) and for all \( x \in \mathbb{R} \). But then there is no value that can be assigned to \( e(x) \) for any \( x \in \mathbb{R} \) and so no identity vector exists. So \( \mathcal{F} \) is not a vector space. \( \square \)

---

**Theorem 3.1**

**Theorem 3.1. Elementary Properties of Vector Spaces.**

Every vector space \( V \) satisfies:

1. the vector \( \bar{0} \) is the unique additive identity in a vector space.
2. if \( \bar{u} + \bar{v} = \bar{u} + \bar{w} \) then \( \bar{v} = \bar{w} \).
3. if \( \bar{u} + \bar{v} = \bar{u} + \bar{w} \) then \( \bar{v} = \bar{w} \).

**Proof.** 1. Suppose that there are two additive identities, \( \bar{0} \) and \( \bar{0}' \). Then consider:

\[
\bar{0} = \bar{0} + \bar{0}' \text{ (since } \bar{0}' \text{ is an additive identity)}
\]

\[
\bar{0}' = \bar{0}' \text{ (since } \bar{0} \text{ is an additive identity)}.
\]

Therefore, \( \bar{0} = \bar{0}' \) and the additive identity is unique.
Theorem 3.1. Elementary Properties of Vector Spaces.

Every vector space $V$ satisfies:
1. The vector $\mathbf{0}$ is the unique additive identity in a vector space,
2. If $u + v = u + w$ then $v = w$,
3. If $u + v = u + w$ then $v = w$.

Proof (continued).
3. Suppose $u + v = u + w$. Then we add $-u$ to both sides of the equation and we get:

$$
(u + v) + (-u) = (u + w) + (-u)
$$

Then the middle side becomes:

$$
(v + u) + (-u) = (w + u) + (-u) \text{ by commutativity, A2}
$$

$$
\tilde{v} + (u - u) = \tilde{w} + (u - u) \text{ by associativity, A1}
$$

$$
\tilde{v} + \mathbf{0} = \tilde{w} + \mathbf{0} \text{ by additive inverse, A4}
$$

$$
\tilde{v} = \tilde{w} \text{ by additive identity, A3}.
$$

The conclusion holds.

Page 190 Number 24

Page 190 Number 24. Let $V$ be a vector space and let $\tilde{v}$ and $\tilde{w}$ be nonzero vectors in $V$. Prove that if $\tilde{v}$ is not a scalar multiple of $\tilde{w}$, then $\tilde{v}$ is not a scalar multiple of $\tilde{v} + \tilde{w}$.

Proof. We consider the (logically equivalent) contrapositive of the claim: If $\tilde{v}$ is a scalar multiple of $\tilde{w}$, then $\tilde{v}$ is a scalar multiple of $\tilde{v} + \tilde{w}$. We prove this and then the original claim follows.

Suppose $\tilde{v}$ is a scalar multiple of $\tilde{v} + \tilde{w}$, say $\tilde{v} = r(\tilde{v} + \tilde{w})$ where $r \in \mathbb{R}$ is a scalar. Then $\tilde{v} = r\tilde{v} + r\tilde{w}$ by S1 and so $\tilde{v} - r\tilde{v} = (r\tilde{v} + r\tilde{w}) - r\tilde{v}$ or

$$
(1-r)\tilde{v} = -r\tilde{v} + (r\tilde{v} + r\tilde{w}) \text{ by S2 and A2}
$$

$$
= (-r\tilde{v} + r\tilde{v}) + r\tilde{w} \text{ by A1}
$$

$$
= \tilde{0} + r\tilde{w} \text{ by A4}
$$

$$
= r\tilde{w} \text{ by A3. (*)}
$$

If $r = 1$ then $0\tilde{v} = 1\tilde{w}$ or $\tilde{0} = \tilde{w}$ (by S4 and Theorem 3.1(4)), but $\tilde{w}$ is a nonzero vector by hypothesis, so $r \neq 1$.

Page 190 Number 24 (continued)

Page 190 Number 24. Let $V$ be a vector space and let $\tilde{v}$ and $\tilde{w}$ be nonzero vectors in $V$. Prove that if $\tilde{v}$ is not a scalar multiple of $\tilde{w}$, then $\tilde{v}$ is not a scalar multiple of $\tilde{v} + \tilde{w}$.

Proof. Then $(*)$, $(1-r)\tilde{v} = r\tilde{w}$, implies that

$$
\frac{1}{1-r}(1-r)\tilde{v} = \frac{1}{1-r}(r\tilde{w})
$$

or, by S3, $\frac{r}{1-r}\tilde{v}$ or, by S4, $\tilde{v} = \frac{r}{1-r}\tilde{w}$. That is, $\tilde{v}$ is a scalar multiple of $\tilde{w}$, as claimed.

Page 190 Number 26

Page 190 Number 26. Use the universality of function spaces to explain how we can view the Euclidean vector space $\mathbb{R}^{mn}$ and the vector space $M_{m,n}$ of all $m \times n$ matrices as essentially the same vector space with just a different notation for the vectors.

Solution. We saw in the previous note that we can use set $S = \{(1,1), (1,2), \ldots, (1,n), (2,1), (2,2), \ldots, (2,n), (3,1), (3,2), \ldots, (m-1,n), (m,1), (m,2), \ldots, (m,n)\}$ and function $f : S \to \mathbb{R}$ to represent an $m \times n$ matrix as

$$
M_f = \begin{bmatrix}
 f((1,1)) & f((1,2)) & \cdots & f((1,n)) \\
 f((2,1)) & f((2,2)) & \cdots & f((2,n)) \\
 \vdots & \vdots & \ddots & \vdots \\
 f((m,1)) & f((m,2)) & \cdots & f((m,n))
\end{bmatrix}
$$

We can also use function $f : S \to \mathbb{R}$ to represent a vector in $\mathbb{R}^{mn}$ as

$$
\tilde{v} = [f((1,1)), f((1,2)), \ldots, f((m,1)), f((1,2)), f((2,2)), \ldots, f((m,n))].
$$
Solution (continued). For \( k \) with \( 1 \leq k \leq mn \), we can write \( k \) as
\[
k = i + (j - 1)m
\]
for some \( j \) with \( 1 \leq j \leq n \) and some \( i \) with \( 1 \leq i \leq m \)
(this is the “Division Algorithm”). So the \( k \)th component of vector \( \vec{v}_f \)
equals the \((i,j)\) entry of \( M \) (and conversely). When matrix \( M_f \) is
multiplied by a scalar \( r \), the \((i,j)\) entry of \( M \) is \( rf((i,j)) \). When vector \( \vec{v}_f \)
is multiplied by a scalar \( r \), the \( k \)th component of \( r\vec{v}_f \) is \( rf((i,j)) \) where
\[
k = i + (j - 1)m
\]
as above. So scalar multiplication “behaves” in the same
way on \( M_f \) and \( \vec{v}_f \). If matrix \( M_g \) and vector \( \vec{v}_g \) are similarly defined using
function \( g : S \rightarrow \mathbb{R} \) then the \((i,j)\) entry of matrix \( M_f + M_g \) is
\[
f((i,j)) + g((i,j))
\]
The \( k \)th component of \( \vec{v}_f + \vec{v}_g \) is \( f((i,j)) + g((i,j)) \)
where \( k = i + (j - 1)m \) as above. So vector/matrix addition “behaves”
the same way as well. The two basic properties of a vector space are scalar
multiplication and vector addition. Since these are the same (or “behave”
the same) then the vector spaces \( \mathbb{R}^{mn} \) and \( M_{m,n} \) are essentially the same.

\[\square\] Note. We clarify this “essentially the same” idea in the Section 3.3,
“Coordinates of Vectors,” when we define a vector space isomorphism.