Example 3.4.A

Let $F$ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3). Let $a$ be a nonzero scalar and define $T : F \to F$ as $T(f) = af$. Is $T$ a linear transformation?

**Solution.** We use Note 3.4.A. Let $f, g \in F$ and let $r, s \in \mathbb{R}$. Then

\[
T(rf + sg) = a(rf + sg)
\]
\[
= a(rf) + a(sg) \text{ by S1}
\]
\[
= (ar)f + (as)g \text{ by S3}
\]
\[
= (ra)f + (sa)g \text{ by commutivity in } \mathbb{R}
\]
\[
= r(af) + s(ag) \text{ by S3}
\]
\[
= rT(f) + sT(g).
\]

Therefore, yes, $T$ is a linear transformation. \(\square\)

---

Example 3.4.B

Let $F$ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3). Let $a$ be a nonzero scalar and define $T : F \to F$ as $T(f) = af$, as in Example 3.4.A. Describe the kernel of $T$.

**Solution.** Let $f \in \ker(T)$. Then $T(f) = 0$ (where 0 denotes the constant function which is 0 for all $x \in \mathbb{R}$). So $T(f) = af = af(x) = 0(x) = 0$. Since $a \neq 0$ then $f(x) = 0$ for all $x \in \mathbb{R}$. That is, $f(x) = 0(x)$ or $f = 0$. So $\ker(T) = \{0\} = \{0(x)\}$. \(\square\)

---

**Page 214 Example 1.** Let $F$ be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$ (see Example 3.1.3), and let $D$ be its subspace of all differentiable functions. Show that differentiation is a linear transformation of $D$ into $F$.

**Proof.** Let $T : D \to F$ be defined as $T(f) = f'$. Let $f, g \in D$ and let $r \in \mathbb{R}$. Since the derivative of a sum is the sum of the derivatives, then

\[
T(f + g) = (f + g)' = f' + g' = T(f) + T(g).
\]

Since the derivative of a multiple of a function is the multiple times the derivative, then

\[
T(rf) = (rf)' = rf' = rT(f).
\]

Therefore $T$ is linear. \(\square\)
Page 215 Example 3. Let $C_{a,b}$ be the set of all continuous functions mapping $[a, b] \to \mathbb{R}$. Then $C_{a,b}$ is a vector space (based on an argument similar to that which justifies that $C = \{ f \in \mathcal{F} | f$ is continuous $\}$ is a subspace of $\mathcal{F}$, as mentioned in Note 3.2.B). Prove that $T : C_{a,b} \to \mathbb{R}$ defined by $T(f) = \int_a^b f(x) \, dx$ is a linear transformation. Such a transformation which maps functions to real numbers is called a linear functional.

**Proof.** Let $f, g \in C_{a,b}$ and let $r \in \mathbb{R}$ be a scalar. Since the integral of a sum is the sum of the integrals and the integral of a multiple of a function is the multiple of the integral of the function, we have

$$T(f+g) = \int_a^b (f(x)+g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx = T(f)+T(g)$$

and

$$T(rf) = \int_a^b rf(x) \, dx = r \int_a^b f(x) \, dx = rT(f).$$

So by Definition 3.9, “Linear Transformation,” $T$ is a linear transformation. □

---

**Theorem 3.5. Preservation of Zero and Subtraction**

Let $V$ and $V'$ be vectors spaces, and let $T : V \to V'$ be a linear transformation. Then

1. $T(\vec{0}) = \vec{0}'$, and
2. $T(\vec{v}_1 - \vec{v}_2) = T(\vec{v}_1) - T(\vec{v}_2)$, for any vectors $\vec{v}_1$ and $\vec{v}_2$ in $V$.

**Proof.** First,

$$T(\vec{0}) = T(0\vec{0}) \text{ by Theorem 3.1(4),}$$

"Elementary Properties of Vector Spaces"

= $0T(\vec{0}) \text{ by Definition 3.9(2),}$$

"Linear Transformations"

= $\vec{0}'$ by Theorem 3.1(4).

---

**Theorem 3.5. Preservation of Zero and Subtraction (continued)**

Let $V$ and $V'$ be vectors spaces, and let $T : V \to V'$ be a linear transformation. Then

1. $T(\vec{0}) = \vec{0}'$, and
2. $T(\vec{v}_1 - \vec{v}_2) = T(\vec{v}_1) - T(\vec{v}_2)$, for any vectors $\vec{v}_1$ and $\vec{v}_2$ in $V$.

**Proof (continued).** Second,

$$T(\vec{v}_1 - \vec{v}_2) = T(\vec{v}_1 - (1)\vec{v}_2) \text{ by S4}$$

= $T(\vec{v}_1 + (-1)\vec{v}_2) \text{ by Theorem 3.1(6)}$

= $T(\vec{v}_1) + (-1)T(\vec{v}_2) \text{ by Note 3.4A}$

= $T(\vec{v}_1) - T(\vec{v}_2) \text{ by Theorem 3.1(6)}.$

So (1) and (2) hold, as claimed. □
**Theorem 3.6**

**Bases and Linear Transformations.**

Let $T : V \rightarrow V'$ be a linear transformation, and let $B$ be a basis for $V$. For any vector $\vec{v}$ in $V$, the vector $T(\vec{v})$ is uniquely determined by the vectors $T(\vec{b})$ for all $\vec{b} \in B$.

**Proof.** Let $T$ and $\overline{T}$ be two linear transformations such that $T(\vec{b}_i) = \overline{T}(\vec{b}_i)$ for each vector $\vec{b}_i \in B$. Let $\vec{v} \in V$. Then for some scalars $r_1, r_2, \ldots, r_k$ we have $\vec{v} = r_1 \vec{b}_1 + r_2 \vec{b}_2 + \cdots + r_k \vec{b}_k$. Then

$$T(\vec{v}) = T(r_1 \vec{b}_1 + r_2 \vec{b}_2 + \cdots + r_k \vec{b}_k) = r_1 T(\vec{b}_1) + r_2 T(\vec{b}_2) + \cdots + r_k T(\vec{b}_k) \text{ by Note 3.4.A}$$

$$= r_1 \overline{T}(\vec{b}_1) + r_2 \overline{T}(\vec{b}_2) + \cdots + r_k \overline{T}(\vec{b}_k) \text{ by Note 3.4.A}$$

$$= \overline{T}(r_1 \vec{b}_1 + r_2 \vec{b}_2 + \cdots + r_k \vec{b}_k)$$

Therefore $T$ and $\overline{T}$ are the same transformations. □

---

**Corollary 3.4.A**

**One-to-One and Kernel.**

A linear transformation $T$ is one-to-one if and only if $\ker(T) = \{\vec{0}\}$.

**Proof.** Let $T : V \rightarrow V'$ where $V$ and $V'$ are vector spaces.

Let $\ker(T) = \{\vec{0}\}$. Suppose for some $\vec{v}_1, \vec{v}_2 \in V$ we have $T(\vec{v}_1) = T(\vec{v}_2)$. Then $T(\vec{v}_1) - T(\vec{v}_2) = \vec{0}$, and so by Theorem 3.5(2), Preservation of Zero and Subtraction, $T(\vec{v}_1 - \vec{v}_2) = \vec{0}$. That is, $\vec{v}_1 - \vec{v}_2 \in \ker(T) = \{\vec{0}\}$. So it must be that $\vec{v}_1 - \vec{v}_2 = \vec{0}$, or $\vec{v}_1 = \vec{v}_2$, and hence $T$ is one-to-one.

Next, suppose $T$ is one-to-one. Since $T(\vec{0}) = \vec{0}'$ by Theorem 3.5(1), “Preservation of Zero and Subtraction,” then for any nonzero vector $\vec{x} \in V$ we must have that $T(\vec{x}) \neq \vec{0}'$. That is, the only vector in $\ker(T)$ is $\vec{0}$. So $\ker(T) = \{\vec{0}\}$, as claimed. □

---

**Theorem 3.8**

**A linear transformation $T : V \rightarrow V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.**

**Proof.** Assume $T$ is invertible and is not one-to-one. Then by the definition of “one-to-one,” for some $\vec{v}_1 \neq \vec{v}_2$ both in $V$, we have $T(\vec{v}_1) = T(\vec{v}_2) = \vec{v}'$. But then $T(\vec{v}_1) - T(\vec{v}_2) = \vec{0}$, and $\vec{v}_1 - \vec{v}_2 = T^{-1}(\vec{v}')$ which implies that $\vec{v}_1 = \vec{v}_2$, a CONTRADICTION. Therefore if $T$ is invertible then $T$ is one-to-one.

From Definition 3.10, “Invertible Transformation,” if $T$ is invertible then for any $\vec{v}' \in V'$ we must have $T^{-1}(\vec{v}') = \vec{v}$ for some $\vec{v} \in V$. Therefore the image of $\vec{v}$ is $\vec{v}' \in V'$ and $T$ is onto.
Theorem 3.8 (continued 1)

**Theorem 3.8.** A linear transformation $T : V \to V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

**Proof (continued).** Finally, we need to show that if $T$ is one-to-one and onto then it is invertible. Suppose that $T$ is one-to-one and onto $V'$. Since $T$ is onto $V'$, then for each $\tilde{v} \in V'$ we can find $\tilde{v} \in V$ such that $T(\tilde{v}) = \tilde{v}'$ and because $T$ is one-to-one, this vector $\tilde{v} \in V$ is unique (from the definition of “one-to-one” and “onto”). Let $T^{-1} : V' \to V$ be defined by $T^{-1}(\tilde{v}') = \tilde{v}$. Then

$$(T \circ T^{-1})(\tilde{v}') = T(T^{-1}(\tilde{v}')) = T(\tilde{v}) = \tilde{v}'$$

and

$$(T^{-1} \circ T)(\tilde{v}) = T^{-1}(T(\tilde{v})) = T^{-1}(\tilde{v}') = \tilde{v},$$

and so $T \circ T^{-1}$ is the identity map on $V'$ and $T^{-1} \circ T$ is the identity map on $V$.

Example 3.4.C

**Example 3.4.C.** Let $\mathcal{F}$ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3). Let $a$ be a nonzero scalar and define $T : \mathcal{F} \to \mathcal{F}$ as $T(f) = af$, as in Example 3.4.A. Determine if $T$ is invertible. If so, find its inverse.

**Solution.** Since $\ker(T) = \{0\}$ by Example 3.4.B, then $T$ is one-to-one by Corollary 3.4.A. For any $g \in \mathcal{F}$, for $f = g/a$ we have $T(f) = T(g/a) = a(g/a) = g$ and so $T$ is onto. So by Theorem 3.8, $T$ is invertible. In fact, $T^{-1}(f) = f/a$ since

$T^{-1}(T(f)) = T^{-1}(af) = (af)/a = f = a(f/a) = T(f/a) = T(T^{-1}(f))$

for all $f \in \mathcal{F}$.

Theorem 3.10. Matrix Representations of Linear Transformations.

**Theorem 3.10.** Let $V$ and $V'$ be finite-dimensional vector spaces and let $B = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)$ and $B' = (\tilde{b}'_1, \tilde{b}'_2, \ldots, \tilde{b}'_m)$ be ordered bases for $V$ and $V'$, respectively. Let $T : V \to V'$ be a linear transformation, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation such that for each $\tilde{v} \in V$, we have $T(\tilde{v}_B) = T(\tilde{v})_{B'}$. Then the standard matrix representation of $T$ is the matrix $A$ whose $j$th column vector is $T(\tilde{b}_j)_{B'}$, and $T(\tilde{v})_{B'} = A\tilde{v}_B$ for all vectors $\tilde{v} \in V$.

**Proof.** Since $B$ is a basis for $V$ and $B$ has $n$ elements, then $\dim(V) = n$ and so by Theorem 3.3.A, “Fundamental Theorem of Finite Dimensional Vector Spaces,” there is isomorphism $\alpha : V \to \mathbb{R}^n$ between $V$ and $\mathbb{R}^n$ where $\alpha(\tilde{v}) = \tilde{v}_B$, as shown in the proof of Theorem 3.3.A.

We need to show that for all $\tilde{v} \in V$ that $T(\tilde{v})_{B'} = A\tilde{v}_B$. We are given that $T(\tilde{v}_B) = T(\tilde{v})_{B'}$, or equivalently

$$\bar{T}(\alpha(\tilde{v})) = T(\tilde{v})_{B'}.$$
Theorem 3.10. Matrix Representations of Linear Transformations.

Let $V$ and $V'$ be finite-dimensional vector spaces and let $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \ldots, \vec{b}'_m)$ be ordered bases for $V$ and $V'$, respectively. Let $T : V \to V'$ be a linear transformation, and let $\overline{T} : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation such that for each $\vec{v} \in V$, we have $\overline{T}(\vec{v}) = T(\vec{v})_{B'}$. Then the standard matrix representation of $\overline{T}$ is the matrix $A$ whose $j$th column vector is $T(\vec{e}_j)_{B'}$, and $T(\vec{v})_{B'} = A\vec{v}_B$ for all vectors $\vec{v} \in V$.

**Proof (continued).** $\overline{T}(\alpha(\vec{v})) = T(\vec{v})_{B'}$. ($\star$)

So we need to show that $\overline{T}(\vec{v}_B) = A(\vec{v}_B)$. Since $\overline{T} : \mathbb{R}^n \to \mathbb{R}^m$, then by Corollary 2.3.A, “Standard Matrix Representation of Linear Transformations,” the standard matrix representation of $\overline{T}$ is the $m \times n$ matrix whose $j$th column is $\overline{T}(\vec{e}_j)$. By the definition of $\alpha$, $\alpha(\vec{b}_j) = \vec{e}_j$, so $\overline{T}(\vec{b}_j) = \overline{T}(\alpha(\vec{b}_j)) = T(\vec{b}_j)_{B'}$ by ($\star$). That is, the $j$th column of $A$ is $T(\vec{b}_j)_{B'}$, as claimed. □

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**Page 227 Number 18**

Let $V$ and $V'$ be vector spaces with ordered bases $B = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \vec{b}'_3, \vec{b}'_4)$, respectively. Let $T : V \to V'$ be the linear transformation having matrix representation $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ relative to $B, B'$. Find $T(\vec{v})$ for $\vec{v} = 3\vec{b}_3 - \vec{b}_1$.

**Solution.** We use Theorem 3.10, “Matrix Representation of Linear Transformations.” Notice that $\vec{v}_B = [-1, 0, 3]$. So $T(\vec{v})_{B'} = A\vec{v}_B = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 3 \\ 7 \end{bmatrix}$.

So $T(\vec{v}) = -7\vec{b}'_1 - 2\vec{b}'_2 + 3\vec{b}'_3 + 7\vec{b}'_4$. □

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**Page 227 Number 22**

Let $T : P_3 \to P_3$ be defined by $T(p(x)) = xD(p(x)) = xp'(x)$ and let the ordered bases $B$ and $B'$ for $P_3$ both be $(x^3, x^2, x, 1)$.

(a) Find the matrix representation $A$ of $T$ relative to $B, B'$.

(b) Working with the matrix $A$ and coordinate vectors, find all solutions $p(x)$ of $T(p(x)) = x^3 - 3x^2 + 4x$.

**Solution.** (a) We use Theorem 3.10, “Matrix Representation of Linear Transformations,” and see that the columns of $A$ are $T(\vec{b}_1)_{B'}$, $T(\vec{b}_2)_{B'}$, $T(\vec{b}_3)_{B'}$, $T(\vec{b}_4)_{B'}$. We find

$T(\vec{b}_1)_{B'} = T(x^3)_{B'} = (x^3)_{B'} = (3x^2)_{B'} = [3, 0, 0, 0]^T$

$T(\vec{b}_2)_{B'} = T(x^2)_{B'} = (x^2)_{B'} = (2x)_{B'} = [0, 2, 0, 0]^T$

$T(\vec{b}_3)_{B'} = T(x)_{B'} = (x)_{B'} = (1)_{B'} = [0, 0, 1, 0]^T$

$T(\vec{b}_4)_{B'} = T(1)_{B'} = (0)_{B'} = [0, 0, 0, 0]^T$.

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**Page 227 Number 22 (continued 1)**

(b) First $(x^3 - 3x^2 + 4x)_{B'} = [1, -3, 4, 0]^T$. From Theorem 3.10, $T(p(x))_{B'} = A\vec{v}_B$, so we want $\vec{v}_B \in \mathbb{R}^4$ such that $A\vec{v}_B = T(p(x))_{B'} = [1, -3, 4, 0]^T$. Let $\vec{v}_B = [v_1, v_2, v_3, v_4]^T$, and consider the augmented matrix for $A\vec{v}_B = [1, -3, 4, 0]^T$:

$\begin{bmatrix} 3 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

We see that this is already in row reduced echelon form and so we need

$3v_1 = 1 \quad v_1 = 1/3$

$2v_2 = -3 \quad v_2 = -3/2$

$v_3 = 4 \quad v_3 = 4$

$v_4 = 0 \quad v_4 = v_4$
Page 227 Number 22. Let $T: \mathcal{P}_3 \to \mathcal{P}_3$ be defined by

$$T(p(x)) = xD(p(x)) = xp'(x)$$

and let the ordered bases $B$ and $B'$ for $\mathcal{P}_3$ both be $(x^3, x^2, x, 1)$.

(a) Find the matrix representation $A$ of $T$ relative to $B, B'$.

(b) Working with the matrix $A$ and coordinate vectors, find all solutions $p(x)$ of $T(p(x)) = x^3 - 3x^2 + 4x$.

Solution. So we take $k = 4$ as a free variable. Then

$$\vec{v}_B = [1/3, -3/2, 4, k]$$

for any $k \in \mathbb{R}$. So $\vec{v} \in \mathcal{P}_3$ is of the form

$$\frac{1}{3}x^3 - \frac{3}{2}x^2 + 4x + k$$

for $k \in \mathbb{R}$. □

Page 227 Number 24. Let $T: \mathcal{P}_3 \to \mathcal{P}_2$ be defined by

$$T(p(x)) = p'(x)|_{2x+1} = p'(2x+1),$$

where $p'(x) = D(p(x))$, and let $B = (b_1, b_2, b_3, b_4) = (x^3, x^2, x, 1)$ and $B' = (x^2, x, 1) = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3)$.

(a) Find the matrix representation $A$ of $T$ relative to $B, B'$.

(b) Use $A$ from part (a) to compute $T(4x^3 - 5x^2 + 4x - 7)$.

Solution. (a) Again we use Theorem 3.10 and find $T(\tilde{b}_1)_{B'}$, $T(\tilde{b}_2)_{B'}$, $T(\tilde{b}_3)_{B'}$, $T(\tilde{b}_4)_{B'}$. First we need the derivatives of $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4$: $\frac{d}{dx}[\tilde{b}_1] = \frac{d}{dx}[x^3] = 3x^2$, $\frac{d}{dx}[	ilde{b}_2] = \frac{d}{dx}[x^2] = 2x$, $\frac{d}{dx}[\tilde{b}_3] = \frac{d}{dx}[x] = 1$, and $\frac{d}{dx}[\tilde{b}_4] = \frac{d}{dx}[1] = 0$. Since $T$ first takes a derivative and then evaluates it at $2x+1$, we have

$$T(x^3) = 3(2x+1)^2 = 12x^2 + 12x + 3, \quad T(x^2) = 2(2x+1) = 4x + 2,$$

and hence

$$T(\tilde{b}_1)_{B'} = T(x^3)_{B'} = (12x^2 + 12x + 3)_{B'} = [12, 12, 3]^T,$n

$$T(\tilde{b}_2)_{B'} = T(x^2)_{B'} = (4x + 2)_{B'} = [0, 4, 2]^T,$n

$$T(\tilde{b}_3)_{B'} = T(x)_{B'} = (1)_{B'} = [0, 0, 1]^T,$n

and

$$T(\tilde{b}_4)_{B'} = T(1)_{B'} = 0_{B'} = [0, 0, 0]^T.$$n

(b) We know from Theorem 3.10, “Matrix Representations of Linear Transformations,” that $T(4x^3 - 5x^2 + 4x - 7)_{B'} = A\vec{v}_B$. Now $\vec{v}_B = [4, -5, 4, -7]$ so

$$T(4x^3 - 5x^2 + 4x - 7)_{B'} = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 12 & 4 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 4 \\ -7 \end{bmatrix} = \begin{bmatrix} 48 \\ 28 \\ 6 \end{bmatrix}$$

and hence

$$T(4x^3 - 5x^2 + 4x - 7) = (48)x^2 + (28)x + (6) = 48x^2 + 28x + 6.$$
Page 228 Number 28

Let \( W = \text{sp}(e^{2x}, e^{4x}, e^{8x}) \) be a subspace of \( F \) (see Example 3.1.3) and let \( B = B' = (e^{2x}, e^{4x}, e^{8x}) \).

(a) Find the matrix representation \( A \) relative to \( B, B' \) of the linear transformation \( T: W \to W \) defined by \( T(f) = \int_{-\infty}^{\infty} f(t) \, dt \).

(b) Find \( A^{-1} \) where \( A \) is the matrix of part (a) and use it to find \( T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x}) \).

Solution. (a) We use Theorem 3.10 and find \( T(\vec{b}_1)_{B'}, T(\vec{b}_2)_{B'}, T(\vec{b}_3)_{B'} \). We have

\[
T(\vec{b}_1) = T(e^{2x}) = \int_{-\infty}^{\infty} e^{2t} \, dt = \lim_{a \to -\infty} \left( \int_{a}^{\infty} e^{2t} \, dt \right) = \lim_{a \to -\infty} \left( \left[ \frac{1}{2} e^{2t} \right]_a^{\infty} \right) = \lim_{a \to -\infty} \left( \frac{1}{2} e^{2a} - \frac{1}{2} e^{2a} \right) = \frac{1}{2} e^{2x} - 0 = \frac{1}{2} e^{2x}
\]

So \( T(\vec{b}_1)_{B'} = \left[ \frac{1}{2}, 0, 0 \right] \), \( T(\vec{b}_2)_{B'} = [0, 1/4, 0] \), \( T(\vec{b}_3)_{B'} = [0, 0, 1/8] \). So

\[
A = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/8 \end{bmatrix}
\]

(b) It is easy to see that \( A^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix} \). By Theorem 3.4.B, \( A^{-1} \) is the matrix representation of \( T^{-1} \) relative to \( B', B \).

(b) Find \( A^{-1} \) where \( A \) is the matrix of part (a) and use it to find \( T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x}) \).

Solution (continued). ...
Page 229 Number 44

Page 229 Number 44. Denote the set of all linear transformations from $V$ to $V'$ as $L(V, V')$. Let $T \in L(V, V')$ and let $r \in \mathbb{R}$ be a scalar. Define $rT : V \to V'$ as $(rT)(\mathbf{v}) = r(T(\mathbf{v}))$ for each $\mathbf{v} \in V$. Prove that $rT \in L(V, V')$.

Solution. Let $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $s, t \in \mathbb{R}$ be scalars. Then

$$(rT)(s\mathbf{v}_1 + t\mathbf{v}_2) = r(T(s\mathbf{v}_1 + t\mathbf{v}_2)) \text{ by the definition of } rT$$

$$= r(sT(\mathbf{v}_1) + tT(\mathbf{v}_2)) \text{ by Note 3.4.A since } T \text{ is linear}$$

$$= r(sT(\mathbf{v}_1)) + r(tT(\mathbf{v}_2)) \text{ by S1}$$

$$= (rs)T(\mathbf{v}_1) + (rt)T(\mathbf{v}_2) \text{ by S3}$$

$$= (sr)T(\mathbf{v}_1) + (tr)T(\mathbf{v}_2) \text{ since multiplication is commutative in } \mathbb{R}$$

$$= s(rT(\mathbf{v}_1)) + t(rT(\mathbf{v}_2)) \text{ by S3}$$

$$= s(rT)(\mathbf{v}_1) + t(rT)(\mathbf{v}_2) \text{ by definition of } rT.$$