Chapter 3. Vector Spaces
Section 3.4. Linear Transformations—Proofs of Theorems
Example 3.4.A

Example 3.4.A. Let $\mathcal{F}$ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3). Let $a$ be a nonzero scalar and define $T : \mathcal{F} \to \mathcal{F}$ as $T(f) = af$. Is $T$ a linear transformation?

Solution. We use Note 3.4.A. Let $f, g \in \mathcal{F}$ and let $r, s \in \mathbb{R}$. Then

$$T(rf + sg) = a(rf + sg)$$
$$= a(rf) + a(sg) \text{ by S1}$$
$$= (ar)f + (as)g \text{ by S3}$$
$$= (ra)f + (sa)g \text{ by commutivity in } \mathbb{R}$$
$$= r(af) + s(ag) \text{ by S3}$$
$$= rT(f) + sT(g).$$

Therefore, yes, $T$ is a linear transformation. □
Example 3.4.A. Let $\mathcal{F}$ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3). Let $a$ be a nonzero scalar and define $T : \mathcal{F} \to \mathcal{F}$ as $T(f) = af$. Is $T$ a linear transformation?

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T(rf + sg) = a(rf + sg) \\
= a(rf) + a(sg) \text{ by } \text{S1} \\
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= r(af) + s(ag) \text{ by } \text{S3} \\
= rT(f) + sT(g).
\]

Therefore, yes, $T$ is a linear transformation. \[\Box\]
Example 3.4.B. Let \( \mathcal{F} \) be the vector space of all functions mapping \( \mathbb{R} \) into \( \mathbb{R} \) (see Example 3.1.3). Let \( a \) be a nonzero scalar and define \( T : \mathcal{F} \to \mathcal{F} \) as \( T(f) = af \), as in Example 3.4.A. Describe the kernel of \( T \).

Solution. Let \( f \in \ker(T) \). Then \( T(f) = 0 \) (where \( 0 = 0(x) \) denotes the constant function which is 0 for all \( x \in \mathbb{R} \)). So \( T(f) = af = af(x) = 0(x) = 0 \). Since \( a \neq 0 \) then \( f(x) = 0 \) for all \( x \in \mathbb{R} \). That is, \( f(x) = 0(x) \) or \( f = 0 \). So \( \ker(T) = \{ 0 \} = \{ 0(x) \} \). □
Example 3.4.B. Let $\mathcal{F}$ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3). Let $a$ be a nonzero scalar and define $T : \mathcal{F} \rightarrow \mathcal{F}$ as $T(f) = af$, as in Example 3.4.A. Describe the kernel of $T$.

Solution. Let $f \in \ker(T)$. Then $T(f) = 0$ (where $0 = 0(x)$ denotes the constant function which is 0 for all $x \in \mathbb{R}$). So $T(f) = af = af(x) = 0(x) = 0$. Since $a \neq 0$ then $f(x) = 0$ for all $x \in \mathbb{R}$. That is, $f(x) = 0(x)$ or $f = 0$. So $\ker(T) = \{0\} = \{0(x)\}$. $\square$
Let $F$ be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$ (see Example 3.1.3), and let $D$ be its subspace of all differentiable functions. Show that differentiation is a linear transformation of $D$ into $F$.

**Proof.** Let $T : D \to F$ be defined as $T(f) = f'$. Let $f, g \in D$ and let $r \in \mathbb{R}$. Since the derivative of a sum is the sum of the derivatives, then

$$T(f + g) = (f + g)' = f' + g' = T(f) + T(g).$$
Page 214 Example 1. Let $F$ be the vector space of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ (see Example 3.1.3), and let $D$ be its subspace of all differentiable functions. Show that differentiation is a linear transformation of $D$ into $F$.

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$$T(f + g) = (f + g)' = f' + g' = T(f) + T(g).$$

Since the derivative of a multiple of a function is the multiple times the derivative, then

$$T(rf) = (rf)' = rf' = rT(f).$$

Therefore $T$ is linear.
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Therefore $T$ is linear.
Page 215 Example 3

Page 215 Example 3. Let $C_{a,b}$ be the set of all continuous functions mapping $[a, b] \to \mathbb{R}$. Then $C_{a,b}$ is a vector space (based on an argument similar to that which justifies that $C = \{f \in \mathcal{F} \mid f$ is continuous$\}$ is a subspace of $\mathcal{F}$, as mentioned in Note 3.2.B). Prove that $T : C_{a,b} \to \mathbb{R}$ defined by $T(f) = \int_{a}^{b} f(x) \, dx$ is a linear transformation. Such a transformation which maps functions to real numbers is called a linear functional.

Proof. Let $f, g \in C_{a,b}$ and let $r \in \mathbb{R}$ be a scalar. Since the integral of a sum is the sum of the integrals and the integral of a multiple of a function is the multiple of the integral of the function, we have

$$T(f+g) = \int_{a}^{b} (f(x)+g(x)) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx = T(f) + T(g)$$
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$$T(f+g) = \int_a^b (f(x)+g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx = T(f) + T(g)$$

and $T(rf) = \int_a^b rf(x) \, dx = r \int_a^b f(x) \, dx = rT(f)$. So, by Definition 3.9, “Linear Transformation,” $T$ is a linear transformation.
Page 215 Example 3

**Page 215 Example 3.** Let \( C_{a,b} \) be the set of all continuous functions mapping \([a, b] \to \mathbb{R}\). Then \( C_{a,b} \) is a vector space (based on an argument similar to that which justifies that \( C = \{ f \in \mathcal{F} \mid f \text{ is continuous} \} \) is a subspace of \( \mathcal{F} \), as mentioned in Note 3.2.B). Prove that \( T : C_{a,b} \to \mathbb{R} \) defined by \( T(f) = \int_a^b f(x) \, dx \) is a linear transformation. Such a transformation which maps functions to real numbers is called a **linear functional**.

**Proof.** Let \( f, g \in C_{a,b} \) and let \( r \in \mathbb{R} \) be a scalar. Since the integral of a sum is the sum of the integrals and the integral of a multiple of a function is the multiple of the integral of the function, we have

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T(f + g) = \int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx = T(f) + T(g)
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and \( T(rf) = \int_a^b rf(x) \, dx = r \int_a^b f(x) \, dx = rT(f) \). So, by Definition 3.9, “Linear Transformation,” \( T \) is a linear transformation. \( \square \)
Page 215 Example 4

**Page 215 Example 4.** Let $C$ be the vector space of all continuous functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Note 3.2.A). Let $a \in \mathbb{R}$ and let $T_a : C \to C$ be defined by $T_a(f) = \int_a^x f(t) \, dt$. Prove that $T$ is a linear transformation.

**Proof.** Similar to the previous example, for $f, g \in C$ and for scalar $r \in \mathbb{R}$ we have

$$T_a(f+g) = \int_a^x (f(t)+g(t)) \, dt = \int_a^x f(t) \, dt + \int_a^x g(t) \, dt = T_a(f) + T_a(g).$$
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So by Definition 3.9, “Linear Transformation,” $T_a$ is a linear transformation.
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**Proof.** Similar to the previous example, for $f, g \in C$ and for scalar $r \in \mathbb{R}$ we have

$$T_a(f + g) = \int_a^x (f(t) + g(t)) \, dt = \int_a^x f(t) \, dt + \int_a^x g(t) \, dt = T_a(f) + T_a(g)$$

and

$$T_a(rf) = \int_a^x rf(t) \, dt = r \int_a^x f(t) \, dt = rT_a(f).$$

So by Definition 3.9, “Linear Transformation,” $T_a$ is a linear transformation.
Theorem 3.5. Preservation of Zero and Subtraction

Let $V$ and $V'$ be vector spaces, and let $T : V \rightarrow V'$ be a linear transformation. Then

1. $T(\vec{0}) = \vec{0}'$, and
2. $T(\vec{v}_1 - \vec{v}_2) = T(\vec{v}_1) - T(\vec{v}_2)$, for any vectors $\vec{v}_1$ and $\vec{v}_2$ in $V$.

Proof. First,

$$T(\vec{0}) = T(0\vec{0}) \text{ by Theorem 3.1(4),}$$

"Elementary Properties of Vector Spaces"

$$= 0T(\vec{0}) \text{ by Definition 3.9(2),}$$

"Linear Transformations"

$$= \vec{0}' \text{ by Theorem 3.1(4).}$$
Theorem 3.5. Preservation of Zero and Subtraction

Let \( V \) and \( V' \) be vector spaces, and let \( T : V \rightarrow V' \) be a linear transformation. Then

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Proof. First,

\[
T(\vec{0}) = T(0\vec{0}) \text{ by Theorem 3.1(4),}
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“Elementary Properties of Vector Spaces”

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Theorem 3.5. Preservation of Zero and Subtraction

Let \( V \) and \( V' \) be vectors spaces, and let \( T : V \to V' \) be a linear transformation. Then

1. \( T(\vec{0}) = \vec{0}' \), and
2. \( T(\vec{v}_1 - \vec{v}_2) = T(\vec{v}_1) - T(\vec{v}_2) \), for any vectors \( \vec{v}_1 \) and \( \vec{v}_2 \) in \( V \).

Proof (continued). Second,

\[
T(\vec{v}_1 - \vec{v}_2) = T(\vec{v}_1 - (1)\vec{v}_2) \text{ by S4} \\
= T(\vec{v}_1 + (-1)\vec{v}_2) \text{ by Theorem 3.1(6)} \\
= T(\vec{v}_1) + (-1)T(\vec{v}_2) \text{ by Note 3.4.A} \\
= T(\vec{v}_1) - T(\vec{v}_2) \text{ by Theorem 3.1(6)}. 
\]

So (1) and (2) hold, as claimed. \(\square\)
Theorem 3.6. Bases and Linear Transformations.

Let $T : V \rightarrow V'$ be a linear transformation, and let $B$ be a basis for $V$. For any vector $\vec{v}$ in $V$, the vector $T(\vec{v})$ is uniquely determined by the vectors $T(\vec{b})$ for all $\vec{b} \in B$.

**Proof.** Let $T$ and $\overline{T}$ be two linear transformations such that $T(\vec{b}_i) = \overline{T}(\vec{b}_i)$ for each vector $\vec{b}_i \in B$. Let $\vec{v} \in V$. Then for some scalars $r_1, r_2, \ldots, r_k$ we have $\vec{v} = r_1 \vec{b}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{b}_k$. 
Theorem 3.6

Theorem 3.6. Bases and Linear Transformations.
Let \( T : V \rightarrow V' \) be a linear transformation, and let \( B \) be a basis for \( V \). For any vector \( \vec{v} \) in \( V \), the vector \( T(\vec{v}) \) is uniquely determined by the vectors \( T(\vec{b}) \) for all \( \vec{b} \in B \).

**Proof.** Let \( T \) and \( \bar{T} \) be two linear transformations such that \( T(\vec{b}_i) = \bar{T}(\vec{b}_i) \) for each vector \( \vec{b}_i \in B \). Let \( \vec{v} \in V \). Then for some scalars \( r_1, r_2, \ldots, r_k \) we have \( \vec{v} = r_1 \vec{b}_1 + r_2 \vec{b}_2 + \cdots + r_k \vec{b}_k \). Then

\[
T(\vec{v}) = T(r_1 \vec{b}_1 + r_2 \vec{b}_2 + \cdots + r_k \vec{b}_k) \\
= r_1 T(\vec{b}_1) + r_2 T(\vec{b}_2) + \cdots + r_k T(\vec{b}_k) \quad \text{by Note 3.4.A} \\
= r_1 \bar{T}(\vec{b}_1) + r_2 \bar{T}(\vec{b}_2) + \cdots + r_k \bar{T}(\vec{b}_k) \\
= \bar{T}(r_1 \vec{b}_1 + r_2 \vec{b}_2 + \cdots + r_k \vec{b}_k) \quad \text{by Note 3.4.A} \\
= \bar{T}(\vec{v}).
\]

Therefore \( T \) and \( \bar{T} \) are the same transformations. \( \square \)
Theorem 3.6. Bases and Linear Transformations.

Let $T : V \rightarrow V'$ be a linear transformation, and let $B$ be a basis for $V$. For any vector $\vec{v}$ in $V$, the vector $T(\vec{v})$ is uniquely determined by the vectors $T(\vec{b})$ for all $\vec{b} \in B$.

**Proof.** Let $T$ and $\overline{T}$ be two linear transformations such that $T(\vec{b}_i) = \overline{T}(\vec{b}_i)$ for each vector $\vec{b}_i \in B$. Let $\vec{v} \in V$. Then for some scalars $r_1, r_2, \ldots, r_k$ we have $\vec{v} = r_1 \vec{b}_1 + r_2 \vec{b}_2 + \cdots + r_k \vec{b}_k$. Then

\[
T(\vec{v}) = T(r_1 \vec{b}_1 + r_2 \vec{b}_2 + \cdots + r_k \vec{b}_k) \\
= r_1 T(\vec{b}_1) + r_2 T(\vec{b}_2) + \cdots + r_k T(\vec{b}_k) \text{ by Note 3.4.A} \\
= r_1 \overline{T}(\vec{b}_1) + r_2 \overline{T}(\vec{b}_2) + \cdots + r_k \overline{T}(\vec{b}_k) \\
= \overline{T}(r_1 \vec{b}_1 + r_2 \vec{b}_2 + \cdots + r_k \vec{b}_k) \text{ by Note 3.4.A} \\
= \overline{T}(\vec{v}).
\]

Therefore $T$ and $\overline{T}$ are the same transformations.
Theorem 3.4.A

**Theorem 3.4.A.** (Page 229 number 46) Let $T : V \rightarrow V'$ be a linear transformation and let $T(\vec{p}) = \vec{b}$ for a particular vector $\vec{p}$ in $V$. The solution set of $T(\vec{x}) = \vec{b}$ is the set $\{\vec{p} + \vec{h} | \vec{h} \in \ker(T)\}$.

**Proof.** Let $\vec{p}$ be a solution of $T(\vec{v}) = \vec{b}$. Then $T(\vec{p}) = \vec{b}$. Let $\vec{h}$ be a solution of $T(\vec{x}) = \vec{0}'$. Then $T(\vec{h}) = \vec{0}'$. 
Theorem 3.4.A

**Theorem 3.4.A.** (Page 229 number 46) Let $T : V \rightarrow V'$ be a linear transformation and let $T(\vec{p}) = \vec{b}$ for a particular vector $\vec{p}$ in $V$. The solution set of $T(\vec{x}) = \vec{b}$ is the set $\{\vec{p} + \vec{h} \mid \vec{h} \in \ker(T)\}$.

**Proof.** Let $\vec{p}$ be a solution of $T(\vec{v}) = \vec{b}$. Then $T(\vec{p}) = \vec{b}$. Let $\vec{h}$ be a solution of $T(\vec{x}) = 0'$. Then $T(\vec{h}) = 0'$. Therefore, by Definition 3.9(1), “Linear Transformation,”

$$T(\vec{p} + \vec{h}) = T(\vec{p}) + T(\vec{h}) = \vec{b} + 0' = \vec{b},$$

and so $\vec{p} + \vec{h}$ is indeed a solution. Also, if $\vec{q}$ is any solution of $T(\vec{x}) = \vec{b}$ then by Theorem 3.5(2), “Preservation of Zero and Subtraction,”

$$T(\vec{q} - \vec{p}) = T(\vec{q}) - T(\vec{p}) = \vec{b} - \vec{b} = 0',$$

and so $\vec{q} - \vec{p}$ is in the kernel of $T$. 
Theorem 3.4.A. (Page 229 number 46) Let $T : \mathbf{V} \rightarrow \mathbf{V}'$ be a linear transformation and let $T(\vec{p}) = \vec{b}$ for a particular vector $\vec{p}$ in $\mathbf{V}$. The solution set of $T(\vec{x}) = \vec{b}$ is the set $\{\vec{p} + \vec{h} \mid \vec{h} \in \text{ker}(T)\}$.

**Proof.** Let $\vec{p}$ be a solution of $T(\vec{v}) = \vec{b}$. Then $T(\vec{p}) = \vec{b}$. Let $\vec{h}$ be a solution of $T(\vec{x}) = 0'$. Then $T(\vec{h}) = 0'$. Therefore, by Definition 3.9(1), “Linear Transformation,”

$$T(\vec{p} + \vec{h}) = T(\vec{p}) + T(\vec{h}) = \vec{b} + 0' = \vec{b},$$

and so $\vec{p} + \vec{h}$ is indeed a solution. Also, if $\vec{q}$ is any solution of $T(\vec{x}) = \vec{b}$ then by Theorem 3.5(2), “Preservation of Zero and Subtraction,”

$$T(\vec{q} - \vec{p}) = T(\vec{q}) - T(\vec{p}) = \vec{b} - \vec{b} = 0',$$

and so $\vec{q} - \vec{p}$ is in the kernel of $T$. Therefore for some $\vec{h} \in \text{ker}(T)$, we have $\vec{q} - \vec{p} = \vec{h}$, for $\vec{q} = \vec{p} + \vec{h}$. 

\[\square\]
Theorem 3.4.A

Theorem 3.4.A. (Page 229 number 46) Let $T : V \rightarrow V'$ be a linear transformation and let $T(\vec{p}) = \vec{b}$ for a particular vector $\vec{p}$ in $V$. The solution set of $T(\vec{x}) = \vec{b}$ is the set $\{\vec{p} + \vec{h} | \vec{h} \in \ker(T)\}$.

Proof. Let $\vec{p}$ be a solution of $T(\vec{v}) = \vec{b}$. Then $T(\vec{p}) = \vec{b}$. Let $\vec{h}$ be a solution of $T(\vec{x}) = 0'$. Then $T(\vec{h}) = 0'$. Therefore, by Definition 3.9(1), “Linear Transformation,”

$$T(\vec{p} + \vec{h}) = T(\vec{p}) + T(\vec{h}) = \vec{b} + 0' = \vec{b},$$

and so $\vec{p} + \vec{h}$ is indeed a solution. Also, if $\vec{q}$ is any solution of $T(\vec{x}) = \vec{b}$ then by Theorem 3.5(2), “Preservation of Zero and Subtraction,”

$$T(\vec{q} - \vec{p}) = T(\vec{q}) - T(\vec{p}) = \vec{b} - \vec{b} = 0',$$

and so $\vec{q} - \vec{p}$ is in the kernel of $T$. Therefore for some $\vec{h} \in \ker(T)$, we have $\vec{q} - \vec{p} = \vec{h}$, for $\vec{q} = \vec{p} + \vec{h}$. 

\[\square\]
Corollary 3.4.A. One-to-One and Kernel.
A linear transformation $T$ is one-to-one if and only if $\ker(T) = \{\vec{0}\}$.

Proof. Let $T : V \rightarrow V'$ where $V$ and $V'$ are vector spaces.

Let $\ker(T) = \{\vec{0}\}$. Suppose for some $\vec{v}_1, \vec{v}_2 \in V$ we have $T(\vec{v}_1) = T(\vec{v}_2)$. Then $T(\vec{v}_1) - T(\vec{v}_2) = \vec{0}'$ and so by Theorem 3.5(2), Preservation of Zero and Subtraction, $T(\vec{v}_1 - \vec{v}_2) = \vec{0}'$. That is, $\vec{v}_1 - \vec{v}_2 \in \ker(T) = \{\vec{0}\}$. So it must be that $\vec{v}_1 - \vec{v}_2 = \vec{0}$, or $\vec{v}_1 = \vec{v}_2$, and hence $T$ is one-to-one.
Corollary 3.4.A. One-to-One and Kernel.
A linear transformation $T$ is one-to-one if and only if $\text{ker}(T) = \{\vec{0}\}$.

Proof. Let $T : V \rightarrow V'$ where $V$ and $V'$ are vector spaces.

Let $\text{ker}(T) = \{\vec{0}\}$. Suppose for some $\vec{v}_1, \vec{v}_2 \in V$ we have $T(\vec{v}_1) = T(\vec{v}_2)$. Then $T(\vec{v}_1) - T(\vec{v}_2) = \vec{0}'$ and so by Theorem 3.5(2), Preservation of Zero and Subtraction, $T(\vec{v}_1 - \vec{v}_2) = \vec{0}'$. That is, $\vec{v}_1 - \vec{v}_2 \in \text{ker}(T) = \{\vec{0}\}$. So it must be that $\vec{v}_1 - \vec{v}_2 = \vec{0}$, or $\vec{v}_1 = \vec{v}_2$, and hence $T$ is one-to-one.

Next, suppose $T$ is one-to-one. Since $T(\vec{0}) = \vec{0}'$ by Theorem 3.5(1), “Preservation of Zero and Subtraction,” then for any nonzero vector $\vec{x} \in V$ we must have that $T(\vec{x}) \neq \vec{0}'$. That is, the only vector in $\text{ker}(T)$ is $\vec{0}$. So $\text{ker}(T) = \{\vec{0}\}$, as claimed.
Corollary 3.4.A. One-to-One and Kernel.
A linear transformation $T$ is one-to-one if and only if $\ker(T) = \{\vec{0}\}$.

**Proof.** Let $T : V \rightarrow V'$ where $V$ and $V'$ are vector spaces.

Let $\ker(T) = \{\vec{0}\}$. Suppose for some $\vec{v}_1, \vec{v}_2 \in V$ we have $T(\vec{v}_1) = T(\vec{v}_2)$. Then $T(\vec{v}_1) - T(\vec{v}_2) = \vec{0}'$ and so by Theorem 3.5(2), Preservation of Zero and Subtraction, $T(\vec{v}_1 - \vec{v}_2) = \vec{0}'$. That is, $\vec{v}_1 - \vec{v}_2 \in \ker(T) = \{\vec{0}\}$. So it must be that $\vec{v}_1 - \vec{v}_2 = \vec{0}$, or $\vec{v}_1 = \vec{v}_2$, and hence $T$ is one-to-one.

Next, suppose $T$ is one-to-one. Since $T(\vec{0}) = \vec{0}'$ by Theorem 3.5(1), “Preservation of Zero and Subtraction,” then for any nonzero vector $\vec{x} \in V$ we must have that $T(\vec{x}) \neq \vec{0}'$. That is, the only vector in $\ker(T)$ is $\vec{0}$. So $\ker(T) = \{\vec{0}\}$, as claimed.
Theorem 3.8. A linear transformation $T : V \rightarrow V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

Proof. ASSUME $T$ is invertible and is not one-to-one. Then by the definition of “one-to-one,” for some $\vec{v}_1 \neq \vec{v}_2$ both in $V$, we have $T(\vec{v}_1) = T(\vec{v}_2) = \vec{v}'$. 
Theorem 3.8

**Theorem 3.8.** A linear transformation $T : V \rightarrow V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

**Proof.** ASSUME $T$ is invertible and is not one-to-one. Then by the definition of “one-to-one,” for some $\vec{v}_1 \neq \vec{v}_2$ both in $V$, we have $T(\vec{v}_1) = T(\vec{v}_2) = \vec{v}'$. But then $\vec{v}_1 = I \vec{v}_1 = T^{-1} \circ T(\vec{v}_1) = T^{-1}(\vec{v}')$ and $\vec{v}_2 = I \vec{v}_2 = T^{-1} \circ T(\vec{v}_2) = T^{-1}(\vec{v}')$, which implies that $\vec{v}_1 = \vec{v}_2$, a CONTRADICTION. Therefore if $T$ is invertible then $T$ is one-to-one.
Theorem 3.8. A linear transformation $T : V \to V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

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From Definition 3.10, “Invertible Transformation,” if $T$ is invertible then for any $\vec{v}' \in V'$ we must have $T^{-1}(\vec{v}') = \vec{v}$ for some $\vec{v} \in V$. Therefore the image of $\vec{v}$ is $\vec{v}' \in V'$ and $T$ is onto.
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From Definition 3.10, “Invertible Transformation,” if $T$ is invertible then for any $\vec{v}' \in V'$ we must have $T^{-1}(\vec{v}') = \vec{v}$ for some $\vec{v} \in V$. Therefore the image of $\vec{v}$ is $\vec{v}' \in V'$ and $T$ is onto.
Theorem 3.8 (continued 1)

**Theorem 3.8.** A linear transformation $T : V \rightarrow V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

**Proof (continued).** Finally, we need to show that if $T$ is one-to-one and onto then it is invertible. Suppose that $T$ is one-to-one and onto $V'$.

Since $T$ is onto $V'$, then for each $\vec{v}' \in V'$ we can find $\vec{v} \in V$ such that $T(\vec{v}) = \vec{v}'$ and because $T$ is one-to-one, this vector $\vec{v} \in V$ is unique (from the definition of “one-to-one” and “onto”).
Theorem 3.8 (continued 1)

**Theorem 3.8.** A linear transformation $T : V \to V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

**Proof (continued).** Finally, we need to show that if $T$ is one-to-one and onto then it is invertible. Suppose that $T$ is one-to-one and onto $V'$. Since $T$ is onto $V'$, then for each $\vec{v}' \in V'$ we can find $\vec{v} \in V$ such that $T(\vec{v}) = \vec{v}'$ and because $T$ is one-to-one, this vector $\vec{v} \in V$ is unique (from the definition of “one-to-one” and “onto”). Let $T^{-1} : V' \to V$ be defined by $T^{-1}(\vec{v}') = \vec{v}$. Then

$$(T \circ T^{-1})(\vec{v}') = T(T^{-1}(\vec{v}')) = T(\vec{v}) = \vec{v}'$$

and

$$ (T^{-1} \circ T)(\vec{v}) = T^{-1}(T(\vec{v})) = T^{-1}(\vec{v}') = \vec{v}, $$

and so $T \circ T^{-1}$ is the identity map on $V'$ and $T^{-1} \circ T$ is the identity map on $V$. 
Theorem 3.8 (continued 1)

**Theorem 3.8.** A linear transformation \( T : V \to V' \) is invertible if and only if it is one-to-one and onto \( V' \). When \( T^{-1} \) exists, it is linear.

**Proof (continued).** Finally, we need to show that if \( T \) is one-to-one and onto then it is invertible. Suppose that \( T \) is one-to-one and onto \( V' \). Since \( T \) is onto \( V' \), then for each \( \vec{v}' \in V' \) we can find \( \vec{v} \in V \) such that \( T(\vec{v}) = \vec{v}' \) and because \( T \) is one-to-one, this vector \( \vec{v} \in V \) is unique (from the definition of “one-to-one” and “onto”). Let \( T^{-1} : V' \to V \) be defined by \( T^{-1}(\vec{v}') = \vec{v} \). Then

\[
(T \circ T^{-1})(\vec{v}') = T(T^{-1}(\vec{v}')) = T(\vec{v}) = \vec{v}'
\]

and

\[
(T^{-1} \circ T)(\vec{v}) = T^{-1}(T(\vec{v})) = T^{-1}(\vec{v}') = \vec{v},
\]

and so \( T \circ T^{-1} \) is the identity map on \( V' \) and \( T^{-1} \circ T \) is the identity map on \( V \).
Theorem 3.8 (continued 2)

**Theorem 3.8.** A linear transformation $T : V \rightarrow V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

**Proof (continued).** Now we need only show that $T^{-1}$ is linear. Suppose $T(\vec{v}_1) = \vec{v}'_1$ and $T(\vec{v}_2) = \vec{v}'_2$; that is, $\vec{v}_1 = T^{-1}(\vec{v}'_1)$ and $\vec{v}_2 = T^{-1}(\vec{v}'_2)$. Then

$$T^{-1}(\vec{v}'_1 + \vec{v}'_2) = T^{-1}(T(\vec{v}_1) + T(\vec{v}_2)) = I(\vec{v}_1 + \vec{v}_2) = \vec{v}_1 + \vec{v}_2.$$
Theorem 3.8 (continued 2)

**Theorem 3.8.** A linear transformation $T : V \to V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

**Proof (continued).** Now we need only show that $T^{-1}$ is linear. Suppose $T(\vec{v}_1) = \vec{v}_1'$ and $T(\vec{v}_2) = \vec{v}_2'$; that is, $\vec{v}_1 = T^{-1}(\vec{v}_1')$ and $\vec{v}_2 = T^{-1}(\vec{v}_2')$.

Then

$$T^{-1}(\vec{v}_1' + \vec{v}_2') = T^{-1}(T(\vec{v}_1) + T(\vec{v}_2))$$
$$= T^{-1}(T(\vec{v}_1 + \vec{v}_2)) \text{ since } T \text{ is linear}$$
$$= (T^{-1} \circ T)(\vec{v}_1 + \vec{v}_2) = I(\vec{v}_1 + \vec{v}_2) = \vec{v}_1 + \vec{v}_2$$
$$= T^{-1}(\vec{v}_1') + T^{-1}(\vec{v}_2').$$
Theorem 3.8 (continued 2)

Theorem 3.8. A linear transformation $T : V \rightarrow V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

Proof (continued). Now we need only show that $T^{-1}$ is linear. Suppose $T(\vec{v}_1) = \vec{v}_1'$ and $T(\vec{v}_2) = \vec{v}_2'$; that is, $\vec{v}_1 = T^{-1}(\vec{v}_1')$ and $\vec{v}_2 = T^{-1}(\vec{v}_2')$. Then

$$T^{-1}(\vec{v}_1' + \vec{v}_2') = T^{-1}(T(\vec{v}_1) + T(\vec{v}_2))$$
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$$= T^{-1}(\vec{v}_1') + T^{-1}(\vec{v}_2').$$

Also (since $T$ is linear)

$$T^{-1}(r\vec{v}_1') = T^{-1}(rT(\vec{v}_1)) = T^{-1}(T(r\vec{v}_1)) = I(r\vec{v}_1) = r\vec{v}_1 = rT^{-1}(\vec{v}_1').$$

Therefore $T^{-1}$ is linear.
Theorem 3.8. A linear transformation $T : V \rightarrow V'$ is invertible if and only if it is one-to-one and onto $V'$. When $T^{-1}$ exists, it is linear.

Proof (continued). Now we need only show that $T^{-1}$ is linear. Suppose $T(\vec{v}_1) = \vec{v}_1'$ and $T(\vec{v}_2) = \vec{v}_2'$; that is, $\vec{v}_1 = T^{-1}(\vec{v}_1')$ and $\vec{v}_2 = T^{-1}(\vec{v}_2')$. Then

$$T^{-1}(\vec{v}_1' + \vec{v}_2') = T^{-1}(T(\vec{v}_1) + T(\vec{v}_2))$$
$$= T^{-1}(T(\vec{v}_1 + \vec{v}_2)) \text{ since } T \text{ is linear}$$
$$= (T^{-1} \circ T)(\vec{v}_1 + \vec{v}_2) = \mathcal{I}(\vec{v}_1 + \vec{v}_2) = \vec{v}_1 + \vec{v}_2$$
$$= T^{-1}(\vec{v}_1') + T^{-1}(\vec{v}_2').$$

Also (since $T$ is linear)

$$T^{-1}(r \vec{v}_1') = T^{-1}(rT(\vec{v}_1)) = T^{-1}(T(r\vec{v}_1)) = \mathcal{I}(r\vec{v}_1) = r\vec{v}_1 = rT^{-1}(\vec{v}_1').$$

Therefore $T^{-1}$ is linear.
Example 3.4.C. Let $\mathcal{F}$ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3). Let $a$ be a nonzero scalar and define $T : \mathcal{F} \to \mathcal{F}$ as $T(f) = af$, as in Example 3.4.A. Determine if $T$ is invertible. If so, find its inverse.

Solution. Since $\ker(T) = \{0\}$ by Example 3.4.B, then $T$ is one-to-one by Corollary 3.4.A. For any $g \in \mathcal{F}$, for $f = g/a$ we have $T(f) = T(g/a) = a(g/a) = g$ and so $T$ is onto.
Example 3.4.C. Let $\mathcal{F}$ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3). Let $a$ be a nonzero scalar and define $T : \mathcal{F} \to \mathcal{F}$ as $T(f) = af$, as in Example 3.4.A. Determine if $T$ is invertible. If so, find its inverse.

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Example 3.4.C. Let $\mathcal{F}$ be the vector space of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ (see Example 3.1.3). Let $a$ be a nonzero scalar and define $T : \mathcal{F} \to \mathcal{F}$ as $T(f) = af$, as in Example 3.4.A. Determine if $T$ is invertible. If so, find its inverse.

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Theorem 3.10

Theorem 3.10. Matrix Representations of Linear Transformations.

Let \( V \) and \( V' \) be finite-dimensional vector spaces and let \( B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n) \) and \( B' = (\vec{b}'_1, \vec{b}'_2, \ldots, \vec{b}'_m) \) be ordered bases for \( V \) and \( V' \), respectively. Let \( T : V \to V' \) be a linear transformation, and let \( \overline{T} : \mathbb{R}^n \to \mathbb{R}^m \) be the linear transformation such that for each \( \vec{v} \in V \), we have \( \overline{T}(\vec{v}_B) = T(\vec{v})_{B'} \). Then the standard matrix representation of \( \overline{T} \) is the matrix \( A \) whose \( j \)-th column vector is \( T(\vec{b}_j)_{B'} \), and \( T(\vec{v})_{B'} = A\vec{v}_B \) for all vectors \( \vec{v} \in V \).

Proof. Since \( B \) is a basis for \( V \) and \( B \) has \( n \) elements, then \( \dim(V) = n \) and so by Theorem 3.3.A, “Fundamental Theorem of Finite Dimensional Vector Spaces,” there is isomorphism \( \alpha : V \to \mathbb{R}^n \) between \( V \) and \( \mathbb{R}^n \) where \( \alpha(\vec{v}) = \vec{v}_B \), as shown in the proof of Theorem 3.3.A.
Theorem 3.10. Matrix Representations of Linear Transformations.

Let $V$ and $V'$ be finite-dimensional vector spaces and let $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \ldots, \vec{b}'_m)$ be ordered bases for $V$ and $V'$, respectively. Let $T : V \to V'$ be a linear transformation, and let $\overline{T} : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation such that for each $\vec{v} \in V$, we have $\overline{T}(\vec{v}_B) = T(\vec{v})_{B'}$. Then the standard matrix representation of $\overline{T}$ is the matrix $A$ whose $j$th column vector is $T(\vec{b}_j)_{B'}$, and $T(\vec{v})_{B'} = A\vec{v}_B$ for all vectors $\vec{v} \in V$.

Proof. Since $B$ is a basis for $V$ and $B$ has $n$ elements, then $\dim(V) = n$ and so by Theorem 3.3.A, “Fundamental Theorem of Finite Dimensional Vector Spaces,” there is isomorphism $\alpha : V \to \mathbb{R}^n$ between $V$ and $\mathbb{R}^n$ where $\alpha(\vec{v}) = \vec{v}_B$, as shown in the proof of Theorem 3.3.A.

We need to show for all $\vec{v} \in V$ that $T(\vec{v})_{B'} = A(\vec{v}_B)$. We are given that $\overline{T}(\vec{v}_B) = T(\vec{v})_{B'}$, or equivalently

$$\overline{T}(\alpha(\vec{v})) = T(\vec{v})_{B'}.$$ (*)
Theorem 3.10. Matrix Representations of Linear Transformations.

Let $V$ and $V'$ be finite-dimensional vector spaces and let $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$ and $B' = (\vec{b}_1', \vec{b}_2', \ldots, \vec{b}_m')$ be ordered bases for $V$ and $V'$, respectively. Let $T : V \to V'$ be a linear transformation, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation such that for each $\vec{v} \in V$, we have $T(\vec{v}_B) = T(\vec{v})_{B'}$. Then the standard matrix representation of $T$ is the matrix $A$ whose $j$th column vector is $T(\vec{b}_j)_{B'}$, and $T(\vec{v})_{B'} = A\vec{v}_B$ for all vectors $\vec{v} \in V$.

**Proof.** Since $B$ is a basis for $V$ and $B$ has $n$ elements, then $\dim(V) = n$ and so by Theorem 3.3.A, “Fundamental Theorem of Finite Dimensional Vector Spaces,” there is isomorphism $\alpha : V \to \mathbb{R}^n$ between $V$ and $\mathbb{R}^n$ where $\alpha(\vec{v}) = \vec{v}_B$, as shown in the proof of Theorem 3.3.A.

We need to show for all $\vec{v} \in V$ that $T(\vec{v})_{B'} = A(\vec{v}_B)$. We are given that $\overline{T}(\vec{v}_B) = T(\vec{v})_{B'}$, or equivalently

$$\overline{T}(\alpha(\vec{v})) = T(\vec{v})_{B'}.$$  \hfill (\ast)
Theorem 3.10. Matrix Representations of Linear Transformations.

Let $V$ and $V'$ be finite-dimensional vector spaces and let $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \ldots, \vec{b}'_m)$ be ordered bases for $V$ and $V'$, respectively. Let $T : V \to V'$ be a linear transformation, and let $\overline{T} : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation such that for each $\vec{v} \in V$, we have $\overline{T}(\vec{v}_B) = T(\vec{v})_{B'}$. Then the standard matrix representation of $\overline{T}$ is the matrix $A$ whose $j$th column vector is $T(\vec{b}_j)_{B'}$, and $T(\vec{v})_{B'} = A\vec{v}_B$ for all vectors $\vec{v} \in V$.

Proof (continued). . . $\overline{T}(\alpha(\vec{v})) = T(\vec{v})_{B'}$. \hspace{1cm} (*)

So we need to show that $\overline{T}(\vec{v}_B) = A(\vec{v}_B)$. Since $\overline{T} : \mathbb{R}^n \to \mathbb{R}^m$, then by Corollary 2.3.A, “Standard Matrix Representation of Linear Transformations,” the standard matrix representation of $\overline{T}$ is the $m \times n$ matrix whose $j$th column is $\overline{T}(\hat{e}_j)$. 
Theorem 3.10. Matrix Representations of Linear Transformations.

Let $V$ and $V'$ be finite-dimensional vector spaces and let $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \ldots, \vec{b}'_m)$ be ordered bases for $V$ and $V'$, respectively. Let $T : V \to V'$ be a linear transformation, and let $\overline{T} : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation such that for each $\vec{v} \in V$, we have $\overline{T}(\vec{v}_B) = T(\vec{v})_{B'}$. Then the standard matrix representation of $\overline{T}$ is the matrix $A$ whose $j$th column vector is $T(\vec{b}_j)_{B'}$, and $T(\vec{v})_{B'} = A\vec{v}_B$ for all vectors $\vec{v} \in V$.

Proof (continued). ... $\overline{T}(\alpha(\vec{v})) = T(\vec{v})_{B'}$. (*)

So we need to show that $\overline{T}(\vec{v}_B) = A(\vec{v}_B)$. Since $\overline{T} : \mathbb{R}^n \to \mathbb{R}^m$, then by Corollary 2.3.A, “Standard Matrix Representation of Linear Transformations,” the standard matrix representation of $\overline{T}$ is the $m \times n$ matrix whose $j$th column is $\overline{T}(\hat{e}_j)$. By the definition of $\alpha$, $\alpha(\vec{b}_j) = \hat{e}_j$, so $\overline{T}(\hat{e}_j) = \overline{T}(\alpha(\vec{b}_j)) = T(\vec{b}_j)_{B'}$ by (*). That is, the $j$th column of $A$ is $T(\vec{b}_j)_{B'}$, as claimed.
Theorem 3.10. Matrix Representations of Linear Transformations.

Let $V$ and $V'$ be finite-dimensional vector spaces and let $B = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \ldots, \vec{b}'_m)$ be ordered bases for $V$ and $V'$, respectively. Let $T : V \to V'$ be a linear transformation, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation such that for each $\vec{v} \in V$, we have $T(\vec{v}_B) = T(\vec{v})_{B'}$. Then the standard matrix representation of $T$ is the matrix $A$ whose $j$th column vector is $T(\vec{b}_j)_{B'}$, and $T(\vec{v})_{B'} = A\vec{v}_B$ for all vectors $\vec{v} \in V$.

Proof (continued). . . . $T(\alpha(\vec{v})) = T(\vec{v})_{B'}$. 

So we need to show that $T(\vec{v}_B) = A(\vec{v}_B)$. Since $T : \mathbb{R}^n \to \mathbb{R}^m$, then by Corollary 2.3.A, “Standard Matrix Representation of Linear Transformations,” the standard matrix representation of $T$ is the $m \times n$ matrix whose $j$th column is $T(\hat{e}_j)$. By the definition of $\alpha$, $\alpha(\vec{b}_j) = \hat{e}_j$, so $T(\hat{e}_j) = T(\alpha(\vec{b}_j)) = T(\vec{b}_j)_{B'}$ by $(\ast)$. That is, the $j$th column of $A$ is $T(\vec{b}_j)_{B'}$, as claimed. □
Let $V$ and $V'$ be vector spaces with ordered bases $B = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \vec{b}'_3, \vec{b}'_4)$, respectively. Let $T : V \to V'$ be the linear transformation having matrix representation $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ relative to $B, B'$. Find $T(\vec{v})$ for $\vec{v} = 3\vec{b}_3 - \vec{b}_1$.

**Solution.** We use Theorem 3.10, “Matrix Representation of Linear Transformations.” Notice that $\vec{v}_B = [-1, 0, 3]$. 

$$
\begin{bmatrix}
-7 \\
-2 \\
3 \\
7
\end{bmatrix}$$

So $T(\vec{v}) = -7\vec{b}'_1 - 2\vec{b}'_2 + 3\vec{b}'_3 + 7\vec{b}'_4$. 

$\square$
Let $V$ and $V'$ be vector spaces with ordered bases $B = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \vec{b}'_3, \vec{b}'_4)$, respectively. Let $T : V \to V'$ be the linear transformation having matrix representation $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ relative to $B, B'$. Find $T(\vec{v})$ for $\vec{v} = 3\vec{b}_3 - \vec{b}_1$.

**Solution.** We use Theorem 3.10, “Matrix Representation of Linear Transformations.” Notice that $\vec{v}_B = [-1, 0, 3]$. So

$$T(\vec{v})_{B'} = A\vec{v}_B = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 3 \\ 7 \end{bmatrix}.$$ 

So $T(\vec{v}) = -7\vec{b}'_1 - 2\vec{b}'_2 + 3\vec{b}'_3 + 7\vec{b}'_4$. $\square$
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Let $V$ and $V'$ be vector spaces with ordered bases $B = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$ and $B' = (\vec{b}'_1, \vec{b}'_2, \vec{b}'_3, \vec{b}'_4)$, respectively. Let $T : V \to V'$ be the linear transformation having matrix representation $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ relative to $B, B'$. Find $T(\vec{v})$ for $\vec{v} = 3\vec{b}_3 - \vec{b}_1$.

Solution. We use Theorem 3.10, “Matrix Representation of Linear Transformations.” Notice that $\vec{v}_B = [-1, 0, 3]$. So

$$T(\vec{v})_{B'} = A\vec{v}_B = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 3 \\ 7 \end{bmatrix}.$$ 

So $T(\vec{v}) = -7\vec{b}'_1 - 2\vec{b}'_2 + 3\vec{b}'_3 + 7\vec{b}'_4$. □
Let $T : \mathcal{P}_3 \rightarrow \mathcal{P}_3$ be defined by $T(p(x)) = xD(p(x)) = xp'(x)$ and let the ordered bases $B$ and $B'$ for $\mathcal{P}_3$ both be $(x^3, x^2, x, 1)$.

(a) Find the matrix representation $A$ of $T$ relative to $B, B'$.

(b) Working with the matrix $A$ and coordinate vectors, find all solutions $p(x)$ of $T(p(x)) = x^3 - 3x^2 + 4x$.

Solution. (a) We use Theorem 3.10, “Matrix Representation of Linear Transformations,” and see that the columns of $A$ are $T(\vec{b}_1)_{B'}, T(\vec{b}_2)_{B'}, T(\vec{b}_3)_{B'}, T(\vec{b}_4)_{B'}$. 
Let $T : \mathcal{P}_3 \rightarrow \mathcal{P}_3$ be defined by $T(p(x)) = xD(p(x)) = xp'(x)$ and let the ordered bases $B$ and $B'$ for $\mathcal{P}_3$ both be $(x^3, x^2, x, 1)$.

(a) Find the matrix representation $A$ of $T$ relative to $B, B'$.

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**Solution.** (a) We use Theorem 3.10, “Matrix Representation of Linear Transformations,” and see that the columns of $A$ are $T(\vec{b}_1)_{B'}, T(\vec{b}_2)_{B'}, T(\vec{b}_3)_{B'}, T(\vec{b}_4)_{B'}$. We find

$$
T(\vec{b}_1)_{B'} = T(x^3)_{B'} = (x(3x^2))_{B'} = (3x^3)_{B'} = [3, 0, 0, 0]^T \\
T(\vec{b}_2)_{B'} = T(x^2)_{B'} = (x(2x))_{B'} = (2x^2)_{B'} = [0, 2, 0, 0]^T \\
T(\vec{b}_3)_{B'} = T(x)_{B'} = (x(1))_{B'} = (x)_{B'} = [0, 0, 1, 0]^T \\
T(\vec{b}_4)_{B'} = T(1)_{B'} = (x(0))_{B'} = (0)_{B'} = [0, 0, 0, 0]^T.
$$
Let $T : \mathcal{P}_3 \rightarrow \mathcal{P}_3$ be defined by

$$T(p(x)) = xD(p(x)) = xp'(x)$$

and let the ordered bases $B$ and $B'$ for $\mathcal{P}_3$ both be $(x^3, x^2, x, 1)$.

(a) Find the matrix representation $A$ of $T$ relative to $B, B'$.
(b) Working with the matrix $A$ and coordinate vectors, find all solutions $p(x)$ of $T(p(x)) = x^3 - 3x^2 + 4x$.

**Solution.** (a) We use Theorem 3.10, “Matrix Representation of Linear Transformations,” and see that the columns of $A$ are $T(\vec{b}_1)_{B'}, T(\vec{b}_2)_{B'}, T(\vec{b}_3)_{B'}, T(\vec{b}_4)_{B'}$. We find

$$
T(\vec{b}_1)_{B'} = T(x^3)_{B'} = (x(3x^2))_{B'} = (3x^3)_{B'} = [3, 0, 0, 0]^T \\
T(\vec{b}_2)_{B'} = T(x^2)_{B'} = (x(2x))_{B'} = (2x^2)_{B'} = [0, 2, 0, 0]^T \\
T(\vec{b}_3)_{B'} = T(x)_{B'} = (x(1))_{B'} = (x)_{B'} = [0, 0, 1, 0]^T \\
T(\vec{b}_4)_{B'} = T(1)_{B'} = (x(0))_{B'} = (0)_{B'} = [0, 0, 0, 0]^T.
$$
Solution. So \( A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).

\[(b)\] First \((x^3 - 3x^2 + 4x)_{B'} = [1, -3, 4, 0]^T\). From Theorem 3.10, \( T(p(x))_{B'} = A\vec{v}_B \), so we want \( \vec{v}_B \in \mathbb{R}^4 \) such that \( A\vec{v}_B = T(p(x))_{B'} = [1, -3, 4, 0]^T \).
Solution. So \( A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).

(b) First \( (x^3 - 3x^2 + 4x)_B' = [1, -3, 4, 0]^T \). From Theorem 3.10, \( T(p(x))_B' = A \vec{v}_B \), so we want \( \vec{v}_B \in \mathbb{R}^4 \) such that \( A \vec{v}_B = T(p(x))_B' = [1, -3, 4, 0]^T \). Let \( \vec{v}_B = [v_1, v_2, v_3, v_4]^T \), and consider the augmented matrix for \( A \vec{v}_B = [1, -3, 4, 0]^T \):

\[
\begin{bmatrix}
3 & 0 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 & -3 \\
0 & 0 & 1 & 0 & 4 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Solution. So \( A = \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \).

(b) First \((x^3 - 3x^2 + 4x)_{B'} = [1, -3, 4, 0]^T\). From Theorem 3.10, \( T(p(x))_{B'} = A\vec{v}_B \), so we want \( \vec{v}_B \in \mathbb{R}^4 \) such that \( A\vec{v}_B = T(p(x))_{B'} = [1, -3, 4, 0]^T \). Let \( \vec{v}_B = [v_1, v_2, v_3, v_4]^T \), and consider the augmented matrix for \( A\vec{v}_B = [1, -3, 4, 0]^T \):

\[
\begin{bmatrix}
3 & 0 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 & -3 \\
0 & 0 & 1 & 0 & 4 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

We see that this is already in row reduced echelon form and so we need

\[
\begin{align*}
3v_1 &= 1 & \Rightarrow & & v_1 &= \frac{1}{3} \\
2v_2 &= -3 & \Rightarrow & & v_2 &= -\frac{3}{2} \\
v_3 &= 4 & \Rightarrow & & v_3 &= 4 \\
0 &= 0 & \Rightarrow & & v_4 &= v_4
\end{align*}
\]
Solution. So \( A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).

(b) First \((x^3 - 3x^2 + 4x)_{B'} = [1, -3, 4, 0]^T\). From Theorem 3.10, \( T(p(x))_{B'} = A\vec{v}_{B'} \), so we want \( \vec{v}_{B'} \in \mathbb{R}^4 \) such that \( A\vec{v}_{B'} = T(p(x))_{B'} = [1, -3, 4, 0]^T \). Let \( \vec{v}_{B'} = [v_1, v_2, v_3, v_4]^T \), and consider the augmented matrix for \( A\vec{v}_{B'} = [1, -3, 4, 0]^T \):

\[
\begin{bmatrix}
3 & 0 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 & -3 \\
0 & 0 & 1 & 0 & 4 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

We see that this is already in row reduced echelon form and so we need

\[
\begin{align*}
3v_1 &= 1 \\
2v_2 &= -3 \\
v_3 &= 4 \\
0 &= 0
\end{align*}
\]

or

\[
\begin{align*}
v_1 &= 1/3 \\
v_2 &= -3/2 \\
v_3 &= 4 \\
v_4 &= v_4
\end{align*}
\]
Page 227 Number 22. Let $T : \mathcal{P}_3 \rightarrow \mathcal{P}_3$ be defined by $T(p(x)) = xD(p(x)) = xp'(x)$ and let the ordered bases $B$ and $B'$ for $\mathcal{P}_3$ both be $(x^3, x^2, x, 1)$.

(a) Find the matrix representation $A$ of $T$ relative to $B, B'$.

(b) Working with the matrix $A$ and coordinate vectors, find all solutions $p(x)$ of $T(p(x)) = x^3 - 3x^2 + 4x$.

Solution. So we take $k = v_4$ as a free variable. Then

$$\vec{v}_B = [1/3, -3/2, 4, k]$$

for any $k \in \mathbb{R}$. So $\vec{v} \in \mathcal{P}_3$ is of the form

\[
\frac{1}{3}x^3 - \frac{3}{2}x^2 + 4x + k \quad \text{for } k \in \mathbb{R}.
\]
Let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be defined by $T(p(x)) = xD(p(x)) = xp'(x)$ and let the ordered bases $B$ and $B'$ for $\mathcal{P}_3$ both be $(x^3, x^2, x, 1)$.

(a) Find the matrix representation $A$ of $T$ relative to $B, B'$.

(b) Working with the matrix $A$ and coordinate vectors, find all solutions $p(x)$ of $T(p(x)) = x^3 - 3x^2 + 4x$.

Solution. So we take $k = v_4$ as a free variable. Then

$\vec{v}_B = [1/3, -3/2, 4, k]$ for any $k \in \mathbb{R}$. So $\vec{v} \in \mathcal{P}_3$ is of the form

$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} x^3 - \begin{bmatrix} 3 \\ 2 \end{bmatrix} x^2 + 4x + k$ for $k \in \mathbb{R}$. □
Let $T : \mathcal{P}_3 \to \mathcal{P}_2$ be defined by $T(p(x)) = p'(x)|_{2x+1} = p'(2x + 1)$, where $p'(x) = D(p(x))$, and let $B = (\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4) = (x^3, x^2, x, 1)$ and $B' = (x^2, x, 1) = (\vec{b}_1', \vec{b}_2', \vec{b}_3')$.

(a) Find the matrix representation $A$ of $T$ relative to $B, B'$.

(b) Use $A$ from part (a) to compute $T(4x^3 - 5x^2 + 4x - 7)$.

Solution. (a) Again we use Theorem 3.10 and find $T(\vec{b}_1)_{B'}$, $T(\vec{b}_2)_{B'}$, $T(\vec{b}_3)_{B'}$, $T(\vec{b}_4)_{B'}$. First we need the derivatives of $\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4$: 

\[
\frac{d}{dx}[\vec{b}_1] = \frac{d}{dx}[x^3] = 3x^2, \quad \frac{d}{dx}[\vec{b}_2] = \frac{d}{dx}[x^2] = 2x, \\
\frac{d}{dx}[\vec{b}_3] = \frac{d}{dx}[x] = 1, \quad \text{and} \quad \frac{d}{dx}[\vec{b}_4] = \frac{d}{dx}[1] = 0.
\]
Let \( T : \mathcal{P}_3 \to \mathcal{P}_2 \) be defined by \( T(p(x)) = p'(x)|_{2x+1} = p'(2x + 1) \), where \( p'(x) = D(p(x)) \), and let \( B = (\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4) = (x^3, x^2, x, 1) \) and \( B' = (x^2, x, 1) = (\vec{b}_1', \vec{b}_2', \vec{b}_3') \).

(a) Find the matrix representation \( A \) of \( T \) relative to \( B, B' \).

(b) Use \( A \) from part (a) to compute \( T(4x^3 - 5x^2 + 4x - 7) \).

**Solution.** (a) Again we use Theorem 3.10 and find \( T(\vec{b}_1)_{B'}, T(\vec{b}_2)_{B'}, T(\vec{b}_3)_{B'}, T(\vec{b}_4)_{B'} \). First we need the derivatives of \( \vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4 \): \( \frac{d}{dx}[\vec{b}_1] = \frac{d}{dx}[x^3] = 3x^2 \), \( \frac{d}{dx}[\vec{b}_2] = \frac{d}{dx}[x^2] = 2x \), \( \frac{d}{dx}[\vec{b}_3] = \frac{d}{dx}[x] = 1 \), and \( \frac{d}{dx}[\vec{b}_4] = \frac{d}{dx}[1] = 0 \). Since \( T \) first takes a derivative and then evaluates it at \( 2x + 1 \), we have

\[
T(x^3) = 3(2x + 1)^2 = 12x^2 + 12x + 3, \quad T(x^2) = 2(2x + 1) = 4x + 2, \\
T(x) = 1, \quad T(1) = 0, \quad \text{and so}
\]

\[
T(\vec{b}_1)_{B'} = T(x^3)_{B'} = (12x^2 + 12x + 3)_{B'} = [12, 12, 3]^T, \\
T(\vec{b}_2)_{B'} = T(x^2)_{B'} = (4x + 2)_{B'} = [0, 4, 2]^T, \\
T(\vec{b}_3)_{B'} = T(x)_{B'} = (1)_{B'} = [0, 0, 1]^T, \quad \text{and}
\]

\[
T(\vec{b}_4)_{B'} = T(1)_{B'} = 0_{B'} = [0, 0, 0]^T.
\]
Let $T : \mathcal{P}_3 \to \mathcal{P}_2$ be defined by $T(p(x)) = p'(x)|_{2x+1} = p'(2x + 1)$, where $p'(x) = D(p(x))$, and let $B = (\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4) = (x^3, x^2, x, 1)$ and $B' = (x^2, x, 1) = (\vec{b}'_1, \vec{b}'_2, \vec{b}'_3)$.

(a) Find the matrix representation $A$ of $T$ relative to $B, B'$.

(b) Use $A$ from part (a) to compute $T(4x^3 - 5x^2 + 4x - 7)$.

Solution. (a) Again we use Theorem 3.10 and find $T(\vec{b}_1)_{B'}$, $T(\vec{b}_2)_{B'}$, $T(\vec{b}_3)_{B'}$, $T(\vec{b}_4)_{B'}$. First we need the derivatives of $\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4$: $\frac{d}{dx}[\vec{b}_1] = \frac{d}{dx}[x^3] = 3x^2$, $\frac{d}{dx}[\vec{b}_2] = \frac{d}{dx}[x^2] = 2x$,

$\frac{d}{dx}[\vec{b}_3] = \frac{d}{dx}[x] = 1$, and $\frac{d}{dx}[\vec{b}_4] = \frac{d}{dx}[1] = 0$. Since $T$ first takes a derivative and then evaluates it at $2x + 1$, we have $T(x^3) = 3(2x + 1)^2 = 12x^2 + 12x + 3$, $T(x^2) = 2(2x + 1) = 4x + 2$, $T(x) = 1$, and $T(1) = 0$, and so $T(\vec{b}_1)_{B'} = T(x^3)_{B'} = (12x^2 + 12x + 3)_{B'} = [12, 12, 3]^T$,

$T(\vec{b}_2)_{B'} = T(x^2)_{B'} = (4x + 2)_{B'} = [0, 4, 2]^T$,

$T(\vec{b}_3)_{B'} = T(x)_{B'} = (1)_{B'} = [0, 0, 1]^T$, and

$T(\vec{b}_4)_{B'} = T(1)_{B'} = 0_{B'} = [0, 0, 0]^T$. 


Solution. So the columns of \( A \) are \( T(\vec{b}_1), T(\vec{b}_2), T(\vec{b}_3), T(\vec{b}_4) \):

\[
A = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 12 & 4 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}.
\]

\( □ \)

(b) We know from Theorem 3.10, “Matrix Representations of Linear Transformations,” that \( T(4x^3 - 5x^2 + 4x - 7)_{B'} = A \vec{v}_B \). Now \( \vec{v}_B = [4, -5, 4, -7] \) so

\[
T(4x^3 - 5x^2 + 4x - 7)_{B'} = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 12 & 4 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 4 \\ -7 \end{bmatrix} = \begin{bmatrix} 48 \\ 28 \\ 6 \end{bmatrix}
\]

and hence

\[
T(4x^3 - 5x^2 + 4x - 7) = (48)x^2 + (28)x + (6)1 = 48x^2 + 28x + 6.
\]
Solution. So the columns of $A$ are $T(\vec{b}_1), T(\vec{b}_2), T(\vec{b}_3), T(\vec{b}_4)$:

$$A = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 12 & 4 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}.$$ □

(b) We know from Theorem 3.10, “Matrix Representations of Linear Transformations,” that $T(4x^3 - 5x^2 + 4x - 7)_{B'} = A\vec{v}_B$. Now $\vec{v}_B = [4, -5, 4, -7]$ so

$$T(4x^3 - 5x^2 + 4x - 7)_{B'} = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 12 & 4 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \\ 4 \\ -7 \end{bmatrix} = \begin{bmatrix} 48 \\ 28 \\ 6 \end{bmatrix}$$

and hence

$$T(4x^3 - 5x^2 + 4x - 7) = (48)x^2 + (28)x + (6)1 = 48x^2 + 28x + 6.$$
Page 227 Number 24. Let $T : \mathcal{P}_3 \rightarrow \mathcal{P}_2$ be defined by

$$T(p(x)) = p'(x)|_{2x+1} = p'(2x + 1),$$

where $p'(x) = D(p(x))$, and let

$$B = (\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4) = (x^3 + x^2, x, 1)$$

and $$B' = (x^2, x, 1) = (\vec{b}'_1, \vec{b}'_2, \vec{b}'_3).$$

(b) Use $A$ from part (a) to compute $T(4x^3 - 5x^2 + 4x - 7)$.

Solution. Notice that

$$\frac{d}{dx}[4x^3 - 5x^2 + 4x - 7] = 12x^2 - 10x + 4$$

and evaluating this at $2x + 1$ gives

$$12(2x + 1)^2 - 10(2x + 1) + 4 = 12(4x^2 + 4x + 1) - 10(2x + 1) + 4$$

$$= 48x^2 + 48x + 12 - 20x - 10 + 4 = 48x^2 + 28x + 6,$$

as expected. □
Let $W = \text{sp}(e^{2x}, e^{4x}, e^{8x})$ be a subspace of $\mathcal{F}$ (see Example 3.1.3) and let $B = B' = (e^{2x}, e^{4x}, e^{8x})$.

(a) Find the matrix representation $A$ relative to $B, B'$ of the linear transformation $T : W \to W$ defined by $T(f) = \int_{-\infty}^{x} f(t) \, dt$.

(b) Find $A^{-1}$ where $A$ is the matrix of part (a) and use it to find $T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x})$.

**Solution.** (a) We use Theorem 3.10 and find $T(\vec{b}_1)_{B'}, T(\vec{b}_2)_{B'}, T(\vec{b}_3)_{B'}$. We have

\[
T(\vec{b}_1) = T(e^{2x}) = \int_{-\infty}^{x} e^{2t} \, dt = \lim_{a \to -\infty} \left( \int_{a}^{x} e^{2t} \, dt \right) = \lim_{a \to -\infty} \left( \left. \left( \frac{1}{2} e^{2t} \right) \right|_{a}^{x} \right)
\]

\[
= \lim_{a \to -\infty} \left( \frac{1}{2} e^{2x} - \frac{1}{2} e^{2a} \right) = \frac{1}{2} e^{2x} - 0 = \frac{1}{2} e^{2x}
\]

..
Let $W = \text{sp}(e^{2x}, e^{4x}, e^{8x})$ be a subspace of $\mathcal{F}$ (see Example 3.1.3) and let $B = B' = (e^{2x}, e^{4x}, e^{8x})$.

(a) Find the matrix representation $A$ relative to $B, B'$ of the linear transformation $T : W \rightarrow W$ defined by $T(f) = \int_{-\infty}^{x} f(t) \, dt$.

(b) Find $A^{-1}$ where $A$ is the matrix of part (a) and use it to find $T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x})$.

Solution. (a) We use Theorem 3.10 and find $T(\vec{b}_1)_{B'}, T(\vec{b}_2)_{B'}, T(\vec{b}_3)_{B'}$. We have

$$T(\vec{b}_1) = T(e^{2x}) = \int_{-\infty}^{x} e^{2t} \, dt = \lim_{a \to -\infty} \left( \int_{a}^{x} e^{2t} \, dt \right) = \lim_{a \to -\infty} \left( \left. \left( \frac{1}{2} e^{2t} \right) \right|^{x}_{a} \right)$$

$$= \lim_{a \to -\infty} \left( \frac{1}{2} e^{2x} - \frac{1}{2} e^{2a} \right) = \frac{1}{2} e^{2x} - 0 = \frac{1}{2} e^{2x}$$
Solution (continued).

\[ T(\vec{b}_2) = T(e^{4x}) = \int_{-\infty}^{x} e^{4t} \, dt = \lim_{a \to -\infty} \left( \int_{a}^{x} e^{4t} \, dt \right) = \lim_{a \to -\infty} \left( \left( \frac{1}{4} e^{4t} \right) \bigg|_{a}^{x} \right) \]

\[ = \lim_{a \to -\infty} \left( \frac{1}{4} e^{4x} - \frac{1}{4} e^{4a} \right) = \frac{1}{4} e^{4x} - 0 = \frac{1}{4} e^{4x} \]

\[ T(\vec{b}_3) = T(e^{8x}) = \int_{-\infty}^{x} e^{8t} \, dt = \lim_{a \to -\infty} \left( \int_{a}^{x} e^{8t} \, dt \right) = \lim_{a \to -\infty} \left( \left( \frac{1}{8} e^{8t} \right) \bigg|_{a}^{x} \right) \]

\[ = \lim_{a \to -\infty} \left( \frac{1}{8} e^{8x} - \frac{1}{8} e^{8a} \right) = \frac{1}{8} e^{8x} - 0 = \frac{1}{8} e^{8x}. \]

So \( T(\vec{b}_1)_{B'} = [1/2, 0, 0], \ T(\vec{b}_2)_{B'} = [0, 1/4, 0], \ T(\vec{b}_3)_{B'} = [0, 0, 1/8]. \) So

\[
A = \begin{bmatrix}
1/2 & 0 & 0 \\
0 & 1/4 & 0 \\
0 & 0 & 1/8 \\
\end{bmatrix}
\]
Page 228 Number 28. Let $W = \text{sp}(e^{2x}, e^{4x}, e^{8x})$ a subspace of $\mathcal{F}$ (see Example 3.1.3) and let $B = B' = (e^{2x}, e^{4x}, e^{8x})$.

(b) Find $A^{-1}$ where $A$ is the matrix of part (a) and use it to find $T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x})$.

Solution (continued). (b) It is easy to see that $A^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$. By Theorem 3.4.B, $A^{-1}$ is the matrix representation of $T^{-1}$ relative to $B', B$.

So by Theorem 3.10, “Matrix Representations of Linear Transformations,” we have that $T^{-1}(\vec{v})_B = A^{-1}\vec{v}_{B'}$ and so

$$T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x})_B = A^{-1}((r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x})'_B) = A^{-1}[r_1, r_2, r_3]^T$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2r_1 \\ 4r_2 \\ 8r_3 \end{bmatrix}.$$
Let $W = \text{sp}(e^{2x}, e^{4x}, e^{8x})$ a subspace of $\mathcal{F}$ (see Example 3.1.3) and let $B = B' = (e^{2x}, e^{4x}, e^{8x})$.

(b) Find $A^{-1}$ where $A$ is the matrix of part (a) and use it to find $T^{-1}(r_1e^{2x} + r_2e^{4x} + r_3e^{8x})$.

Solution (continued). (b) It is easy to see that $A^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$. By Theorem 3.4.B, $A^{-1}$ is the matrix representation of $T^{-1}$ relative to $B'$, $B$. So by Theorem 3.10, “Matrix Representations of Linear Transformations,” we have that $T^{-1}(\vec{v})_B = A^{-1}\vec{v}_{B'}$ and so

$$T^{-1}(r_1e^{2x} + r_2e^{4x} + r_3e^{8x})_B = A^{-1}((r_1e^{2x} + r_2e^{4x} + r_3e^{8x})'_B) = A^{-1}[r_1, r_2, r_3]^T$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 2r_1 \\ 4r_2 \\ 8r_3 \end{bmatrix}.$$
Page 228 Number 28. Let \( W = \text{sp}(e^{2x}, e^{4x}, e^{8x}) \) a subspace of \( \mathcal{F} \) (see Example 3.1.3) and let \( B = B' = (e^{2x}, e^{4x}, e^{8x}) \).

(b) Find \( A^{-1} \) where \( A \) is the matrix of part (a) and use it to find \( T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x}) \).

Solution (continued). . .

\[
T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x})_B = \begin{bmatrix} 2r_1 \\ 4r_2 \\ 8r_3 \end{bmatrix}.
\]

So translating this using basis \( B \) we have

\[
T^{-1}(r_1 e^{2x} + r_2 e^{4x} + r_3 e^{8x}) = 2r_1 e^{2x} + 4r_2 e^{4x} + 8r_3 e^{8x}. \]

□
Denote the set of all linear transformations from $V$ to $V'$ as $L(V, V')$. Let $T \in L(V, V')$ and let $r \in \mathbb{R}$ be a scalar. Define $rT : V \to V'$ as $(rT)\vec{v} = r(T(\vec{v}))$ for each $\vec{v} \in V$. Prove that $rT \in L(V, V')$.

**Solution.** Let $\vec{v}_1, \vec{v}_2 \in V$ and $s, t \in \mathbb{R}$ be scalars.
Denote the set of all linear transformations from $V$ to $V'$ as $L(V, V')$. Let $T \in L(V, V')$ and let $r \in \mathbb{R}$ be a scalar. Define $rT : V \to V'$ as $(rT)\vec{v} = r(T(\vec{v}))$ for each $\vec{v} \in V$. Prove that $rT \in L(V, V')$.

**Solution.** Let $\vec{v}_1, \vec{v}_2 \in V$ and $s, t \in \mathbb{R}$ be scalars. Then

$$(rT)(s\vec{v}_1 + t\vec{v}_2) = r(T(s\vec{v}_1 + t\vec{v}_2)) \text{ by the definition of } rT$$

$$= r(sT(\vec{v}_1) + tT(\vec{v}_2)) \text{ by Note 3.4.A since } T \text{ is linear}$$

$$= r(sT(\vec{v}_1)) + r(tT(\vec{v}_2)) \text{ by S1}$$

$$= (rs)T(\vec{v}_1) + (rt)T(\vec{v}_2) \text{ by S3}$$
Denote the set of all linear transformations from $V$ to $V'$ as $L(V, V')$. Let $T \in L(V, V')$ and let $r \in \mathbb{R}$ be a scalar. Define $rT : V \to V'$ as $(rT)\vec{v} = r(T(\vec{v}))$ for each $\vec{v} \in V$. Prove that $rT \in L(V, V')$.

**Solution.** Let $\vec{v}_1, \vec{v}_2 \in V$ and $s, t \in \mathbb{R}$ be scalars. Then

$$(rT)(s\vec{v}_1 + t\vec{v}_2) = r(T(s\vec{v}_1 + t\vec{v}_2)) \text{ by the definition of } rT$$

$$= r(sT(\vec{v}_1) + tT(\vec{v}_2)) \text{ by Note 3.4.A since } T \text{ is linear}$$

$$= r(sT(\vec{v}_1)) + r(tT(\vec{v}_2)) \text{ by S1}$$

$$= (rs)T(\vec{v}_1) + (rt)T(\vec{v}_2) \text{ by S3}$$

$$= (sr)T(\vec{v}_1) + (tr)T(\vec{v}_2) \text{ since multiplication is commutative in } \mathbb{R}$$

$$= s(rT(\vec{v}_1)) + t(rT(\vec{v}_2)) \text{ by S3}$$

$$= s(rT)(\vec{v}_1) + t(rT)(\vec{v}_2) \text{ by definition of } rT.$$
Denote the set of all linear transformations from $V$ to $V'$ as $L(V, V')$. Let $T \in L(V, V')$ and let $r \in \mathbb{R}$ be a scalar. Define $rT : V \to V'$ as $(rT)\vec{v} = r(T(\vec{v}))$ for each $\vec{v} \in V$. Prove that $rT \in L(V, V')$.

**Solution.** Let $\vec{v}_1, \vec{v}_2 \in V$ and $s, t \in \mathbb{R}$ be scalars. Then

$$(rT)(s\vec{v}_1 + t\vec{v}_2) = r(T(s\vec{v}_1 + t\vec{v}_2)) \text{ by the definition of } rT$$

$$= r(sT(\vec{v}_1) + tT(\vec{v}_2)) \text{ by Note 3.4.A since } T \text{ is linear}$$

$$= r(sT(\vec{v}_1)) + r(tT(\vec{v}_2)) \text{ by S1}$$

$$= (rs)T(\vec{v}_1) + (rt)T(\vec{v}_2) \text{ by S3}$$

$$= (sr)T(\vec{v}_1) + (tr)T(\vec{v}_2) \text{ since multiplication is commutative in } \mathbb{R}$$

$$= s(rT(\vec{v}_1)) + t(rT(\vec{v}_2)) \text{ by S3}$$

$$= s(rT)(\vec{v}_1) + t(rT)(\vec{v}_2) \text{ by definition of } rT.$$
Page 229 Number 44. Denote the set of all linear transformations from $V$ to $V'$ as $L(V, V')$. Let $T \in L(V, V')$ and let $r \in \mathbb{R}$ be a scalar. Define $rT : V \to V'$ as $(rT)(\vec{v}) = r(T(\vec{v}))$ for each $\vec{v} \in V$. Prove that $rT \in L(V, V')$.

Solution (continued). So $rT$ is a linear transformation by Note 3.4.A. □
Page 229 Number 44. Denote the set of all linear transformations from $V$ to $V'$ as $L(V, V')$. Let $T \in L(V, V')$ and let $r \in \mathbb{R}$ be a scalar. Define $rT : V \rightarrow V'$ as $(rT)\vec{v} = r(T(\vec{v}))$ for each $\vec{v} \in V$. Prove that $rT \in L(V, V')$.

Solution (continued). So $rT$ is a linear transformation by Note 3.4.A. □

Note. In Exercise 43 it is shown for $T_1, T_2 \in L(V, V')$ that $T_1 + T_2 \in L(V, V')$ where we define $(T_1 + T_2)(\vec{v}_1 + \vec{v}_2) = T_1(\vec{v}_1) + T_2(\vec{v}_2)$. So $L(V, V')$ is closed under vector addition and scalar multiplication.
Page 229 Number 44. Denote the set of all linear transformations from $V$ to $V'$ as $L(V, V')$. Let $T \in L(V, V')$ and let $r \in \mathbb{R}$ be a scalar. Define $rT : V \rightarrow V'$ as $(rT)\vec{v} = r(T(\vec{v}))$ for each $\vec{v} \in V$. Prove that $rT \in L(V, V')$.

Solution (continued). So $rT$ is a linear transformation by Note 3.4.A. □

Note. In Exercise 43 it is shown for $T_1, T_2 \in L(V, V')$ that $T_1 + T_2 \in L(V, V')$ where we define $(T_1 + T_2)(\vec{v}_1 + \vec{v}_2) = T_1(\vec{v}_1) + T_2(\vec{v}_2)$. So $L(V, V')$ is closed under vector addition and scalar multiplication. Therefore, by Theorem 3.2, “Test for a Subspace,” $L(V, V')$ is a subspace of the vector space of all functions mapping $V$ into $V'$ (see “Summary Item 5 on page 188).
Page 229 Number 44. Denote the set of all linear transformations from $V$ to $V'$ as $L(V, V')$. Let $T \in L(V, V')$ and let $r \in \mathbb{R}$ be a scalar. Define $rT : V \rightarrow V'$ as $(rT)\vec{v} = r(T(\vec{v}))$ for each $\vec{v} \in V$. Prove that $rT \in L(V, V')$.

Solution (continued). So $rT$ is a linear transformation by Note 3.4.A. □

Note. In Exercise 43 it is shown for $T_1, T_2 \in L(V, V')$ that $T_1 + T_2 \in L(V, V')$ where we define $(T_1 + T_2)(\vec{v}_1 + \vec{v}_2) = T_1(\vec{v}_1) + T_2(\vec{v}_2)$. So $L(V, V')$ is closed under vector addition and scalar multiplication. Therefore, by Theorem 3.2, “Test for a Subspace,” $L(V, V')$ is a subspace of the vector space of all functions mapping $V$ into $V'$ (see “Summary Item 5 on page 188).
Page 226 Number 12. Let $D_\infty$ be the vector space of functions mapping $\mathbb{R}$ into $\mathbb{R}$ that have derivatives of all orders. It can be shown that the kernel of a linear transformation $T : D_\infty \rightarrow D_\infty$ of the form $T(f) = a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + a_0 f$, where $a_n \neq 0$, is an $n$-dimensional subspace of $D_\infty$. Use this information to find the solution set in $D_\infty$ of the differential equation $y' - y = x$. HINT: a particular solution to the differential equation is $y = -x - 1$.

Solution. First, we consider the “homogeneous” linear differential equation $y' - y = 0$; that is, $y' = y$. We know from Calculus that if $y' = y$ then $y = k e^x$ for some $k \in \mathbb{R}$ ($y' = y$ is a separable differential equation and can be solved by separation of variables and integration). This is the general solution to $y' - y = 0$ and the set of all such solutions form a subspace of the vector space $\mathcal{F}$ of all real valued functions defined on $\mathbb{R}$ (see exercise 3.2.40).
Page 226 Number 12. Let $D_\infty$ be the vector space of functions mapping $\mathbb{R}$ into $\mathbb{R}$ that have derivatives of all orders. It can be shown that the kernel of a linear transformation $T : D_\infty \to D_\infty$ of the form $T(f) = a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + a_0 f$, where $a_n \neq 0$, is an $n$-dimensional subspace of $D_\infty$. Use this information to find the solution set in $D_\infty$ of the differential equation $y' - y = x$. HINT: a particular solution to the differential equation is $y = -x - 1$.

Solution. First, we consider the “homogeneous” linear differential equation $y' - y = 0$; that is, $y' = y$. We know from Calculus that if $y' = y$ then $y = ke^x$ for some $k \in \mathbb{R}$ ($y' = y$ is a separable differential equation and can be solved by separation of variables and integration). This is the general solution to $y' - y = 0$ and the set of all such solutions form a subspace of the vector space $\mathcal{F}$ of all real valued functions defined on $\mathbb{R}$ (see exercise 3.2.40).
Page 226 Number 12. Let $D_\infty$ be the vector space of functions mapping $\mathbb{R}$ into $\mathbb{R}$ that have derivatives of all orders. It can be shown that the kernel of a linear transformation $T : D_\infty \to D_\infty$ of the form $T(f) = a_nf^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_1f' + a_0f$, where $a_n \neq 0$, is an $n$-dimensional subspace of $D_\infty$. Use this information to find the solution set in $D_\infty$ of the differential equation $y' - y = x$. HINT: a particular solution to the differential equation is $y = -x - 1$.

Solution (continued). By the solution to Exercise 3.2.41, all solutions to $y' - y = x$ are of the form $p(x) + h(x)$ where $p(x)$ is a particular solution to $y' - y = x$ and $h(x)$ is some solution to the homogeneous differential equation $y' - y = 0$. We are given that a particular solution to $y' - y = x$ is $y = -x - 1$. So the solution set to the differential equation $y' - y = x$ is $\{-x - 1 + ke^x \mid k \in \mathbb{R}\}$. □