Solution (continued).

\[
\begin{vmatrix}
3 & 2 & 0 & 0 & 0 \\
-1 & 4 & 1 & 0 & 0 \\
0 & -3 & 5 & 2 & 0 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & -1 & 2 \\
\end{vmatrix} = - \begin{vmatrix}
-1 & 4 & 1 & 0 & 0 \\
3 & 2 & 0 & 0 & 0 \\
0 & -3 & 5 & 2 & 0 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & -1 & 2 \\
\end{vmatrix} \quad \text{Row Exchange: } R_1 \leftrightarrow R_2

\begin{align*}
R_2 & \rightarrow R_2 + 3R_3 \\
R_5 & \rightarrow R_5 + R_4
\end{align*}

= \begin{vmatrix}
-1 & 4 & 1 & 0 & 0 \\
0 & 14 & 3 & 0 & 0 \\
0 & -3 & 5 & 2 & 0 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 6 \\
\end{vmatrix} \quad \text{Row Addition: } R_2 \rightarrow R_2 + 3R_3 \text{ and } R_5 \rightarrow R_5 + R_4

\text{Solution (continued) . . .}

\[
\begin{vmatrix}
-1 & 4 & 1 & 0 & 0 \\
0 & 14 & 3 & 0 & 0 \\
0 & -3 & 5 & 2 & 0 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 6 \\
\end{vmatrix} = - \begin{vmatrix}
-1 & 4 & 1 & 0 & 0 \\
0 & 14 & 3 & 0 & 0 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 6 \\
\end{vmatrix} \quad \text{Row Addition: } R_3 \rightarrow R_3 + (3/14)R_2

= -(-1)(14)(79/14)(1)(6) = 474 \text{ by Example 4.2.4.}
\]
Theorem 4.5. Cramer’s Rule.
Consider the linear system $A\vec{x} = \vec{b}$, where $A = [a_{ij}]$ is an $n \times n$ invertible matrix,

$$
\vec{x} = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
\quad \text{and} \quad
\vec{b} = \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_n
\end{bmatrix}.
$$

The system has a unique solution given by

$$x_k = \frac{\det(B_k)}{\det(A)} \quad \text{for} \quad k = 1, 2, \ldots, n,$$

where $B_k$ is the matrix obtained from $A$ by replacing the $k$th column vector of $A$ by the column vector $\vec{b}$.

**Proof.** Since $A$ is invertible, we know that the linear system $A\vec{x} = \vec{b}$ has a unique solution by Theorem 1.16. Let $\vec{x}$ be this unique solution.

---

**Proof (continued).** That is, $AX_k$ is the matrix $B_k$ described in the statement of the theorem. From the equation $AX_k = B_k$ and Theorem 4.4, ”The Multiplicative Property,” we have

$$\det(A) \det(X_k) = \det(B_k).$$

Computing $\det(X_k)$ by expanding by minors across the $k$th row (applying Theorem 4.2, “General Expansion by Minors”), we see that $\det(X_k) = x_k$ and thus $\det(A)x_k = \det(B_k)$. Because $A$ is invertible, we know that $\det(A) \neq 0$ by Theorem 4.3, “Determinant Criterion for Invertibility,” and so $x_k = \det(B_k)/\det(A)$ as claimed.

---

**Page 272 Number 26.** Use Cramer’s Rule to solve

$$
\begin{align*}
3x_1 + x_2 &= 5 \\
2x_1 + x_2 &= 0
\end{align*}
$$

**Solution.** We have $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$. So $B_1 = \begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 3 & 5 \\ 2 & 0 \end{bmatrix}$. Next, $\det(A) = (3)(1) - (1)(2) = 1$, $\det(B_1) = (5)(1) - (1)(1) = 5$, and $\det(B_2) = (3)(0) - (5)(2) = -10$. So by Cramer’s Rule,

$$
\begin{align*}
x_1 &= \frac{\det(B_1)}{\det(A)} = \frac{5}{1} = 5 \quad \text{and} \quad x_2 &= \frac{\det(B_2)}{\det(A)} = \frac{-10}{1} = -10.
\end{align*}
$$

So $x_1 = 5$ and $x_2 = -10$. □
Page 272 Number 18

Find the adjoint of \( A = \begin{bmatrix} 3 & 0 & 3 \\ 4 & 1 & -2 \\ -5 & 1 & 4 \end{bmatrix} \).

Solution. First, we compute the 9 cofactors:
\[
\begin{aligned}
a'_{11} &= \frac{1}{1} - \frac{2}{4} = 6, \\
a'_{12} &= -\frac{4}{-5} - \frac{2}{4} = -6, \\
a'_{13} &= -\frac{4}{-5} + 1 = 9, \\
a'_{21} &= -\frac{0}{3} + \frac{3}{4} = 3, \\
a'_{22} &= \frac{3}{-5} + \frac{3}{4} = 27, \\
a'_{23} &= -\frac{3}{-5} + 1 = -3, \\
a'_{31} &= -\frac{3}{0} + \frac{3}{-2} = -3, \\
a'_{32} &= -\frac{3}{4} + 0 = 3, \\
a'_{33} &= \frac{3}{4} + 1 = 3,
\end{aligned}
\]
so \( A' = [a'_{ij}] \ldots \)

---

Theorem 4.6

**Property of the Adjoint.**

Let \( A \) be \( n \times n \). Then
\[
(adj(A))A = A(adj(A) = (det(A))I.
\]

**Proof.** Let \( A = [a_{ij}] \). Define \( B \) as the matrix which results from replacing Row \( j \) of \( A \) with Row \( i \) of \( A \). Then, by Theorem 4.2.A, “Properties of Determinants,”
\[
\text{det}(B) = \begin{cases} 
\text{det}(A) & \text{if } i = j \text{ (since } B = A) \\
0 & \text{if } i \neq j, \text{ by Theorem 4.2.A(3), “Equal Row Property.”}
\end{cases}
\]

Now we can expand \( \text{det}(B) \) about the \( j \)th row of \( B \) to get by Theorem 4.2, “General Expansion by Minors,” that \( \text{det}(B) = \sum_{s=1}^{n} a_{is}a'_{js} \) and so
\[
\sum_{s=1}^{n} a_{is}a'_{js} = \begin{cases} 
\text{det}(A) & \text{if } i = j \\
0 & \text{if } i \neq j.
\end{cases}
\]  

(2)

Notice that the \((i, j)\) entry of \( A'(A')^\top \) is \( \sum_{k=1}^{n} a_{ik}a'_{jk} \) where \( A' = [a'_{ij}] \).

---

Page 272 Number 18 (continued)

Find the adjoint of \( A = \begin{bmatrix} 3 & 0 & 3 \\ 4 & 1 & -2 \\ -5 & 1 & 4 \end{bmatrix} \).

Solution (continued). . . \( A' = [a'_{ij}] = \begin{bmatrix} 6 & -6 & 9 \\ 3 & 27 & -3 \\ -3 & 18 & 3 \end{bmatrix} \) and
\[
\text{adj}(A) = (A')^\top = \begin{bmatrix} 6 & 3 & -3 \\ -6 & 27 & 18 \\ 9 & -3 & 3 \end{bmatrix}.
\]

\( \square \)

---

Theorem 4.6 (continued)

**Property of the Adjoint.**

Let \( A \) be \( n \times n \). Then
\[
(adj(A))A = A(adj(A) = (det(A))I.
\]

**Proof (continued).** Since we can express the right-hand side of (2) as \( \text{det}(A)I \), then we have \( A(A')^\top = A(adj(A) = \text{det}(A)I. \)

Similarly if matrix \( C \) results from replacing Column \( i \) of \( A \) with Column \( j \) of \( A \) and by computing \( \text{det}(C) \) by expanding along the \( i \)th column of \( C \) we get
\[
\sum_{i=1}^{n} a_{ij}a'_{ik} = \begin{cases} 
\text{det}(A) & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\]

and so \( (A')^\top A = \text{adj}(A)A = \text{det}(A)I \). Hence, \( \text{adj}(A)A = A(adj(A) = \text{det}(A)I) \), as claimed. \( \square \)
Page 272 Number 18

Page 272 Number 18. Find the inverse of \( A = \begin{bmatrix} 3 & 0 & 3 \\ 4 & 1 & -2 \\ -5 & 1 & 4 \end{bmatrix} \) using \( \text{adj}(A) \).

Solution. First, we compute \( \text{det}(A) \) by expanding along the first row:
\[
\text{det}(A) = (3) \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - (0) + (3) \begin{vmatrix} 1 & 4 \\ 4 & 1 \end{vmatrix} = 3(6) + 3(9) = 45.
\]
So by Corollary 4.3.A, “Formula for \( A^{-1} \),” we have (using \( \text{adj}(A) \) computed above)
\[
A^{-1} = \frac{\text{adj}(A)}{\text{det}(A)} = \frac{1}{45} \begin{bmatrix} 2 & 1 & -1 \\ -6 & 27 & 18 \\ 9 & -3 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{15} & \frac{1}{15} & -\frac{1}{15} \\ -\frac{2}{5} & \frac{9}{5} & \frac{6}{5} \\ \frac{3}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}.
\]

Page 272 Number 22

Page 272 Number 22. Given that \( A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( \text{det}(A^{-1}) = 3 \), find the matrix \( A \).

Solution. We know from Corollary 4.3.A, “Formula for \( A^{-1} \),” that \( A^{-1} = \text{adj}(A)/\text{det}(A) \). Now \( \text{det}(A^{-1}) = 1/\text{det}(A) \) by Exercise 4.2.31, so \( \text{det}(A) = 1/\text{det}(A^{-1}) = 1/3 \). If \( A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \) then \( a_{11}^{-1} = a_{22} \).

\[
a_{12} = -a_{21}, \quad a_{12}' = -a_{12}, \quad \text{and} \quad a_{22}' = a_{11}.
\]
So \( A' = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix} \) and 
\[
\text{adj}(A) = (A')^T = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \text{det}(A)A^{-1} = \frac{1}{3} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \text{so}
\]
\[
a_{11} = d/3, \quad a_{12} = -b/3, \quad a_{21} = -c/3, \quad \text{and} \quad a_{22} = a/3.
\]
Therefore
\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} d/3 & -b/3 \\ -c/3 & a/3 \end{bmatrix}.
\]

Page 273 Number 36

Page 273 Number 36. Prove that the inverse of a nonsingular upper-triangular matrix is upper triangular.

Solution. Let \( A = [a_{ij}] \) be a (square) nonsingular upper triangular matrix; that is, \( a_{ij} = 0 \) for \( i > j \). Now the minor matrix \( A_{ij} \) (obtained from \( A \) by eliminating Row \( i \) and Column \( j \) from \( A \)) is upper triangular for \( i < j \):

Page 273 Number 36 (continued)

Page 273 Number 36 (continued). Then, with \( i < j \), \((n-1) \times (n-1) \) minor matrix \( A_{ij} \) has a 0 in its \((i, i)\) entry (it is element \( a_{i+1,i} = 0 \) in matrix \( A \)). So for \( i < j \), \( A_{ij} \) is upper triangular with a 0 on the diagonal. By Example 4.2.4 (the determinant of an upper triangular square matrix is the product of the diagonal entries), \( \text{det}(A_{ij}) = 0 \) and so cofactor \( a_{ij} = (-1)^{i+j} \text{det}(A_{ij}) = 0 \) for \( i < j \). So matrix \( A' \) has 0 in entry \((i, j)\) whenever \( i < j \). That is, \( A' \) is lower triangular. Hence \( \text{adj}(A) = (A')^T \) is upper triangular. Since \( A \) is nonsingular then by Theorem 4.3, “Determinant Criterion for Invertibility,” \( \text{det}(A) \neq 0 \). By Corollary 4.3.A, “A Formula for the Inverse of an Invertible Matrix,” \( A^{-1} = \frac{1}{\text{det}(A)} \text{adj}(A) \) and so \( A^{-1} \) is also upper triangular. \( \square \)
Page 273 Number 38. Let $A$ be an $n \times n$ nonsingular matrix. Prove that $\det(\text{adj}(A)) = \det(A)^{n-1}$.

Solution. By Corollary 4.3.A, “A Formula for $A^{-1}$,” $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$.

By Exercise 4.2.31, $\det(A^{-1}) = 1/\det(A)$, so we have

$$\frac{1}{\det(A)} = \det(A^{-1}) = \det\left(\frac{1}{\det(A)} \text{adj}(A)\right)$$

$$= \frac{1}{\det(A)^n} \det(\text{adj}(A))$$

by Theorem 4.2.A(4), “Scalar Multiplication Property,” applied to each of the $n$ rows of adj$(A)$.

So $\det(\text{adj}(A)) = \det(A)^n/\det(A) = \det(A)^{n-1}$, as claimed. \qed

Page 273 Number 38. Let $A$ be an $n \times n$ nonsingular matrix. Prove that $\det(\text{adj}(A)) = \det(A)^{n-1}$.

Note. This result also holds if $A$ is an $n \times n$ singular matrix. If $A$ is singular then $\det(A) = 0$ by Theorem 4.3, “Determinant Criterion for Invertibility.” By Exercise 37, $A$ is invertible if and only if adj$(A)$ is invertible. So $\det(A) = 0$ implies $\det(\text{adj}(A)) = 0$ (again, by Theorem 4.3), and so Exercise 38 holds for nonsingular square matrices as well.