Page 336 number 4

Find the projection of \([1, 2, 1]\) on the line with parametric equation \(x = 3t, y = t, z = 2t\) in \(\mathbb{R}^3\).

**Solution.** A line is a translation of a one-dimensional subspace and is of the form \(\vec{x} = \vec{d} t + \vec{a}\) where \(\vec{d}\) is the direction vector and \(\vec{a}\) is a translation vector (see Section 2.5, “Lines, Planes, and Other Flats”). Here, \(\vec{d} = [3, 1, 2]\) and \(\vec{a} = [0, 0, 0]\) so, in fact, the line is not translated and so is a subspace spanned by \(\vec{d} = [3, 1, 2]\). So we apply the previous definition to get the projection \(\vec{p}\) of \(\vec{b} = [1, 2, 1]\) on \(\text{sp}(\vec{d})\):

\[
\vec{p} = \text{proj}_{\vec{d}}(\vec{b}) = \frac{\vec{b} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} = \frac{[1, 2, 1] \cdot [3, 1, 2]}{[3, 1, 2] \cdot [3, 1, 2]} [3, 1, 2]
\]

\[
= \frac{(1)(3) + (2)(1) + (1)(2)}{3^2 + 1^2 + 2^2} [3, 1, 2] = \frac{7}{14} [3, 1, 2] = \left[\frac{3}{2}, 1/2, 1\right].
\]

\(\square\)

Page 336 number 10

Find the orthogonal complement of the plane \(2x + y + 3z = 0\) in \(\mathbb{R}^3\).

**Solution.** A plane is a translation of a two-dimensional space of the form \(\vec{x} = t_1 \vec{d}_1 + t_2 \vec{d}_2 + \vec{a}\) where \(\vec{d}_1\) and \(\vec{d}_2\) form a basis for the two-dimensional space and \(\vec{a}\) is a translation vector (see Section 2.5, “Lines, Planes, and Other Flats”). Here, we can take \(\vec{a} = \vec{0}\) so that the plane is not translated and is in fact a subspace of \(\mathbb{R}^3\). So we just need a basis for the subspace. We pick two linearly independent vectors in the subspace, say \(\vec{d}_1 = [1, -2, 0]\) and \(\vec{d}_2 = [0, -3, 1]\) (though there are infinitely many such choices). Then using the technique described above, we take

\[
A = \begin{bmatrix}
1 & -2 & 0 \\
0 & -3 & 1
\end{bmatrix}
\]

and find the nullspace of \(A\) by considering the system of equations \(Ax = \vec{0}\) (see Note 6.1.A):

\[
\begin{bmatrix}
1 & -2 & 0 \\
0 & -3 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

So we have

\[
x_1 = (2/3)x_3 = 0 \quad \text{or} \quad x_1 = (2/3)x_3
\]

\[
x_2 = (1/3)x_3 = 0 \quad \text{or} \quad x_2 = (1/3)x_3
\]

or with \(x_3 = 3t\) as a free variable, \(x_1 = 2t, x_2 = t,\) and \(x_3 = 3t\). So \(W^\perp\) is the nullspace of \(A\): \(W^\perp = \text{sp}([[2, 1, 3]])\). \(\square\)
Theorem 6.1. Properties of $W^\perp$.
The orthogonal complement $W^\perp$ of a subspace $W$ of $\mathbb{R}^n$ has the following properties:

1. $W^\perp$ is a subspace of $\mathbb{R}^n$.
2. $\dim(W^\perp) = n - \dim(W)$.
3. $(W^\perp)^\perp = W$.
4. Each vector $\vec{b} \in \mathbb{R}^n$ can be expressed uniquely in the form $\vec{b} = \vec{b}_W + \vec{b}_W^\perp$ for $\vec{b}_W \in W$ and $\vec{b}_W^\perp \in W^\perp$.

Proof. Let $\dim(W) = k$, and let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a basis for $W$. Let $A$ be the $k \times n$ matrix having $\vec{v}_i$ as its $i$th row vector for $i = 1, 2, \ldots, k$.

Property (1) follows from the fact that $W^\perp$ is the nullspace of matrix $A$, by Note 6.1.A, and therefore is a subspace of $\mathbb{R}^n$.

Proof (continued). For Property 4, let $\{\vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n\}$ be a basis for $n - k$ dimensional (by Property 2) subspace $W^\perp$. We now show that

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \cup \{\vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n\} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$

is a basis for $\mathbb{R}^n$. Consider the linear combination

$$r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k + s_{k+1}\vec{v}_{k+1} + s_{k+2}\vec{v}_{k+2} + \cdots + s_n\vec{v}_n = \vec{0}. \quad (*)$$

This equation implies

$$r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k = -s_{k+1}\vec{v}_{k+1} - s_{k+2}\vec{v}_{k+2} - \cdots - s_n\vec{v}_n.$$

Proof (continued). Notice that the vector on the left hand side of this equation is in $W$ and the vector on the right hand side is in $W^\perp$. But both sides of the equation represent the same vector (d’uh, it’s an equation!) so both sides of the equation represent a vector in both $W$ and $W^\perp$. So this vector must be orthogonal to itself. The only vector orthogonal to itself is $\vec{0}$ (since $0 = \vec{v} \cdot \vec{v} = ||\vec{v}||^2$ implies $\vec{v} = \vec{0}$). Since the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent and $r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k = \vec{0}$ then we must have $r_1 = r_2 = \cdots = r_k = 0$. Similarly, $\vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n$ are linearly independent and $s_{k+1}\vec{v}_{k+1} + s_{k+2}\vec{v}_{k+2} + \cdots + s_n\vec{v}_n = \vec{0}$ implies $s_{k+1} = s_{k+2} = \cdots = s_n = 0$. From equation $(*)$, we see that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a linearly independent set. Since the set contains $n$ linearly independent vectors in $\mathbb{R}^n$ then $\dim(\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)) = n$ and so by Exercise 2.1.38, $\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) = \mathbb{R}^n$ and so $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$. 

\[ \text{Proof (continued)} \]
**Theorem 6.1 (continued 4)**

**Proof (continued).** So each \( \vec{b} \in \mathbb{R}^n \) can be written as
\[
\vec{b} = (r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k) + (s_{k+1} \vec{v}_{k+1} + s_{k+2} \vec{v}_{k+2} + \cdots + s_n \vec{v}_n),
\]
where
\[
r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k \in W \quad \text{and} \quad s_{k+1} \vec{v}_{k+1} + s_{k+2} \vec{v}_{k+2} + \cdots + s_n \vec{v}_n \in W^\perp.
\]
for unique \( r_1, r_2, \ldots, r_k, s_{k+1}, s_{k+2}, \ldots, s_n \) (by Definition 1.17, “Basis for a Subspace”). So any \( \vec{b} \in \mathbb{R}^n \) can be expressed in the form \( \vec{b} = \vec{b}_W + \vec{b}_{W^\perp} \) where \( \vec{b}_W \in W \) and \( \vec{b}_{W^\perp} \in W^\perp \). Since each vector in \( \mathbb{R}^n \) is a unique linear combination of \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \), then the choice of \( \vec{b}_W \) and \( \vec{b}_{W^\perp} \) are unique.

**Page 336 number 20(b)**

**Page 336 number 20(b).** Find the projection of \( \vec{b} = [-2, 1, 3, -5] \) on to the subspace \( W = \text{sp}(\hat{e}_1, \hat{e}_4) \in \mathbb{R}^4 \).

**Solution.** We are given a basis for \( W = \text{sp}(\hat{e}_1, \hat{e}_4) \), namely \{\( \hat{e}_1, \hat{e}_4 \}\}. Certainly a basis for \( W^\perp \) is given by \{\( \hat{e}_2, \hat{e}_3 \)\}. So we take the ordered basis \( \{\hat{e}_1, \hat{e}_4, \hat{e}_2, \hat{e}_3\} \) of \( \mathbb{R}^4 \) and we have \( \vec{b} = -2\hat{e}_1 - 5\hat{e}_4 + 1\hat{e}_2 + 3\hat{e}_3 \) (and so the coordinate vector \( \vec{r} \) of \( \vec{b} \) relative to the ordered basis \( \{\hat{e}_1, \hat{e}_4, \hat{e}_2, \hat{e}_3\} \) is \( \vec{r} = [-2, -5, 1, 3] \)). Then by the Note 6.1.B, the projection of \( \vec{b} \) on to \( W \) is
\[
\vec{b}_W = \text{proj}_W(\vec{b}) = r_1 \hat{e}_1 + r_2 \hat{e}_4 = -2[1, 0, 0, 0] - 5[0, 0, 0, 1] = [-2, 0, 0, -5].
\]

**Page 335 Example 6**

**Page 335 Example 6.** Consider the inner product space \( \mathcal{P}_{[0,1]} \) of all polynomial functions defined on the interval \([0, 1]\) with inner product
\[
\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x) \, dx.
\]
Find the projection of \( f(x) = x \) on \( \text{sp}(1) \) and then find the projection of \( x \) on \( \text{sp}(1)^\perp \).

**Solution.** We follow the definition of the projection \( \vec{p} \) of \( \vec{b} \) on \( \text{sp}(\vec{a}) \) in \( \mathbb{R}^n \),
\[
\vec{p} = \text{proj}_a(\vec{b}) = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a},
\]
but instead of dot products in \( \mathbb{R}^n \) we use the inner product in \( \mathcal{P}_{[0,1]} \). So the desired projection, with \( \vec{b} = x \) and \( \vec{a} = 1 \), is
\[
\begin{align*}
\langle x, 1 \rangle & = \int_0^1 x \cdot 1 \, dx = \frac{1}{2}x^2|_0^1 = \frac{1}{2}, \\
\langle 1, 1 \rangle & = \int_0^1 1 \cdot 1 \, dx = \int_0^1 1 \, dx = x|_0^1 = 1,
\end{align*}
\]
\[
\langle x, 1 \rangle \langle 1, 1 \rangle = \frac{1}{2} \times 1 = \frac{1}{2}.
\]

**Page 335 Example 6 (continued)**

**Page 335 Example 6.** Consider the inner product space \( \mathcal{P}_{[0,1]} \) of all polynomial functions defined on the interval \([0, 1]\) with inner product
\[
\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x) \, dx.
\]
Find the projection of \( f(x) = x \) on \( \text{sp}(1) \) and then find the projection of \( x \) on \( \text{sp}(1)^\perp \).

**Solution (continued).** Notice that with \( W = \text{sp}(1) \) then we have from Definition 6.2 that \( \vec{b} = \vec{b}_W + \vec{b}_{W^\perp} \) and we can find \( \vec{b}_{W^\perp} \) (where \( W^\perp = \text{sp}(1)^\perp \)) as \( \vec{b}_{W^\perp} = \vec{b} - \vec{b}_W = x - 1/2 \). □
Page 337 number 26. Let $A$ be an $m \times n$ matrix.

(a) Prove that the set $W$ of row vectors $\vec{x}$ in $\mathbb{R}^m$ such that $\vec{x}A = \vec{0}$ is a subspace of $\mathbb{R}^m$.

(b) Prove that the subspace $W$ in part (a) and the column space of $A$ are orthogonal complements in $\mathbb{R}^m$.

**Proof.** (a) We use definition 1.16, “Subspace of $\mathbb{R}^n$.” Let $W = \{\vec{x} \in \mathbb{R}^m \mid \vec{x}A = \vec{0}\}$. We must check $W$ for closure under vector addition and scalar multiplication. Let $\vec{x}_1, \vec{x}_2 \in W$ and let $r$ be a scalar. Then:

\[
(\vec{x}_1 + \vec{x}_2)A = \vec{x}_1A + \vec{x}_2A \text{ by Theorem 1.3.A(10)},
\]

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(here we treat $\vec{x}$ as a matrix)

\[
= \vec{0} + \vec{0} \text{ since } \vec{x}_1, \vec{x}_2 \in W
\]

\[= \vec{0},
\]

(b) Let $A$ be an $m \times n$ matrix. Prove that the subspace $W$ in part (a) and the column space of $A$ are orthogonal complements in $\mathbb{R}^m$.

**Proof.** Recall that by Definition 1.8, “Matrix Product,” the $(i, j)$ entry of the matrix product $AB$ is the dot product of the $i$th row of $A$ with the $j$th column of $B$.

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Page 337 number 26 (continued 2)

**Proof (continued).** So for $\vec{x} \in W$ (here we treat row vector $\vec{x} \in \mathbb{R}^m$ as a $1 \times m$ matrix) we have that $\vec{x}A$ is a $1 \times n$ matrix (or a row vector in $\mathbb{R}^n$) and for $\vec{x} \in W$ we have $\vec{x}A = \vec{0} \in \mathbb{R}^n$. So the $j$th entry of $\vec{x}A = \vec{0}$ is the dot product of $\vec{x}$ with the $j$th column of $A$ and, since $\vec{x}A = \vec{0}$, this dot product must be 0 for each $j = 1, 2, \ldots, n$. So by Definition 1.7, “Perpendicular or Orthogonal Vectors,” each $\vec{x} \in W$ is orthogonal to each column of $A$. Also, by definition, $W$ contain all vectors $\vec{x}$ in $\mathbb{R}^m$ which satisfy $\vec{x}A = \vec{0}$ (i.e., all vectors $\vec{x}$ in $\mathbb{R}^m$ which are perpendicular to all columns of $A$). The column space of $A$ is the span of the columns of $A$ and since $\vec{x} \in W$ is orthogonal to each column of $A$ then $\vec{x}$ is orthogonal to each vector which is in the span of the columns of $A$. Conversely, any vector $\vec{x}$ in the orthogonal complement of the column space of $A$ must be orthogonal to all linear combinations of the columns of $A$; in particular such $\vec{x}$ must be orthogonal to each column of $A$ and hence such $\vec{x}$ is in $W$. So the orthogonal complement of the column space of $A$ is $W$.

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Page 337 number 28

Let $W$ be a subspace of $\mathbb{R}^n$ with orthogonal complement $W^\perp$. Writing $\vec{a} = \vec{a}_W + \vec{a}_{W^\perp}$, as in Theorem 6.1, prove that $\|\vec{a}\| = \sqrt{\|\vec{a}_W\|^2 + \|\vec{a}_{W^\perp}\|^2}$.

**Solution.** By Note 1.2.A, $\|\vec{a}\|^2 = \vec{a} \cdot \vec{a}$, so we have

\[
\|\vec{a}\|^2 = (\vec{a}_W + \vec{a}_{W^\perp}) \cdot (\vec{a}_W + \vec{a}_{W^\perp})
\]

\[
= \vec{a}_W \cdot \vec{a}_W + \vec{a}_W \cdot \vec{a}_{W^\perp} + \vec{a}_{W^\perp} \cdot \vec{a}_W + \vec{a}_{W^\perp} \cdot \vec{a}_{W^\perp}
\]

by Theorem 1.3, “Properties of Dot Products”

\[
= \|\vec{a}_W\|^2 + \|\vec{a}_{W} \cdot \vec{a}_{W^\perp} + \vec{a}_{W^\perp} \cdot \vec{a}_W + \|\vec{a}_{W^\perp}\|^2
\]

by Note 1.2.A

\[
= \|\vec{a}_W\|^2 + 0 + 0 + \|\vec{a}_{W^\perp}\|^2
\]

since $\vec{a}_W$ and $\vec{a}_{W^\perp}$ are orthogonal.

Not taking square roots (and observing that $\|\vec{a}\|$ is nonnegative) gives $\|\vec{a}\| = \sqrt{\|\vec{a}_W\|^2 + \|\vec{a}_{W^\perp}\|^2}$. 

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