Chapter 6: Orthogonality
Section 6.1. Projections—Proofs of Theorems

Linear Algebra

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Find the projection of $[1, 2, 1]$ on the line with parametric equation $x = 3t$, $y = t$, $z = 2t$ in $\mathbb{R}^3$.

Solution. A line is a translation of a one-dimensional subspace and is of the form $\vec{x} = t\vec{d} + \vec{a}$ where $\vec{d}$ is the direction vector and $\vec{a}$ is a translation vector (see Section 2.5, "Lines, Planes, and Other Flats"). Here, $\vec{d} = [3, 1, 2]$ and $\vec{a} = [0, 0, 0]$ so, in fact, the line is not translated and so is a subspace spanned by $\vec{d} = [3, 1, 2]$. So we apply the previous definition to get the projection $\vec{p}$ of $\vec{b} = [1, 2, 1]$ on $\text{sp}(\vec{d})$:

$$\vec{p} = \text{proj}_\vec{d}(\vec{b}) = \frac{\vec{b} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} = \frac{[1, 2, 1] \cdot [3, 1, 2]}{[3, 1, 2] \cdot [3, 1, 2]} [3, 1, 2]$$

$$= \frac{(1)(3) + (2)(1) + (1)(2)}{3^2 + 1^2 + 2^2} [3, 1, 2] = \frac{7}{14} [3, 1, 2] = \frac{1}{2} [3/2, 1/2, 1].$$

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Find the orthogonal complement of the plane $2x + y + 3z = 0$ in $\mathbb{R}^3$.

Solution. A plane is a translation of a two-dimensional space of the form $\vec{x} = t_1 \vec{d}_1 + t_2 \vec{d}_2 + \vec{a}$ where $\vec{d}_1$ and $\vec{d}_2$ form a basis for the two-dimensional space and $\vec{a}$ is a translation vector (see Section 2.5, "Lines, Planes, and Other Flats"). Here, we can take $\vec{a} = \vec{0}$ so that the plane is not translated and is in fact a subspace of $\mathbb{R}^3$. So we just need a basis for the subspace. We pick two linearly independent vectors in the subspace, say $\vec{d}_1 = [1, -2, 0]$ and $\vec{d}_2 = [0, -3, 1]$ (though there are infinitely many such choices). Then using the technique described above, we take $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -3 & 1 \end{bmatrix}$ and find the nullspace of $A$ by considering the system of equations $A\vec{x} = \vec{0}$:

$$[A \mid \vec{0}] = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & -3 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - (2/3)R_2} \begin{bmatrix} 1 & 0 & -2/3 & 0 \\ 0 & -3 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 / (-3)} \begin{bmatrix} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -1/3 & 0 \end{bmatrix}.$$

So we have

$$x_1 = (2/3)x_3 = 0 \quad x_1 = (2/3)x_3$$
$$x_2 = (1/3)x_3 = 0 \quad x_2 = (1/3)x_3$$
$$x_3 = x_3 \quad x_3 = x_3$$

or with $x_3 = 3t$ as a free variable, $x_1 = 2t$, $x_2 = t$, and $x_3 = 3t$. So $W^\perp$ is the nullspace of $A$:

$$W^\perp = \text{sp}([2, 1, 3]).$$
Theorem 6.1

Properties of $W^\perp$.
The orthogonal complement $W^\perp$ of a subspace $W$ of $\mathbb{R}^n$ has the following properties:

1. $W^\perp$ is a subspace of $\mathbb{R}^n$.
2. $\dim(W^\perp) = n - \dim(W)$.
3. $(W^\perp)^\perp = W$.
4. Each vector $\vec{b} \in \mathbb{R}^n$ can be expressed uniquely in the form $\vec{b} = \vec{b}_W + \vec{b}_{W^\perp}$ for $\vec{b}_W \in W$ and $\vec{b}_{W^\perp} \in W^\perp$.

Proof. Let $\dim(W) = k$, and let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a basis for $W$. Let $A$ be the $k \times n$ matrix having $\vec{v}_i$ as its $i$th row vector for $i = 1, 2, \ldots, k$.

Property (1) follows from the fact that $W^\perp$ is the nullspace of matrix $A$ and therefore is a subspace of $\mathbb{R}^n$.

Proof (continued). For Property 2, consider the rank equation of $A$:

$$\text{rank}(A) + \text{nullity}(A) = n.$$ 

Since $\dim(W) = \text{rank}(A)$ and since $W^\perp$ is the nullspace of $A$, then

$$\dim(W^\perp) = n - \dim(W).$$

For Property 3, we have by Property 1 that $W^\perp$ is a subspace of $\mathbb{R}^n$. By Property 2 we have

$$\dim(W^\perp)^\perp = n - \dim(W^\perp) = n - (n - k) = k.$$ 

Since every vector in $W$ is orthogonal to subspace $W^\perp$, then $W$ is a subspace of $(W^\perp)^\perp$ ($(W^\perp)^\perp$ is a subspace of $\mathbb{R}^n$ by two applications of Property 1). Since $W$ and $(W^\perp)^\perp$ have the same dimension then by Exercise 2.138, $W$ must be equal to $(W^\perp)^\perp$.

Proof (continued). For Property 4, let $\{\vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n\}$ be a basis for $n - k$ dimensional (by Property 2) subspace $W^\perp$. We now show that

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \cup \{\vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n\} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$$

is a basis for $\mathbb{R}^n$. Consider the linear combination

$$r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k + s_{k+1}\vec{v}_{k+1} + s_{k+2}\vec{v}_{k+2} + \cdots + s_n\vec{v}_n = \vec{0}. \quad (*)$$

This equation implies

$$r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k = -s_{k+1}\vec{v}_{k+1} - s_{k+2}\vec{v}_{k+2} - \cdots - s_n\vec{v}_n.$$ 

Proof (continued). Notice that the vector on the left hand side of this equation is in $W$ and the vector on the right hand side is in $W^\perp$. But both sides of the equation represent the same vector (d'uh, it's an equation!) so both sides of the equation represent a vector in both $W$ and $W^\perp$. So this vector must be orthogonal to itself. The only vector orthogonal to itself is $\vec{0}$ (since $0 = \vec{v} \cdot \vec{v} = \|\vec{v}\|^2$ implies $\vec{v} = \vec{0}$). Since the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent and

$$r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k = \vec{0}$$

then we must have $r_1 = r_2 = \cdots = r_k = 0$. Similarly, $\vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n$ are linearly independent and

$$s_{k+1}\vec{v}_{k+1} + s_{k+2}\vec{v}_{k+2} + \cdots + s_n\vec{v}_n = \vec{0}$$

implies $s_{k+1} = s_{k+2} = \cdots = s_n = 0$. From equation $(*)$, we see that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a linearly independent set. Since the set contains $n$ linearly independent vectors in $\mathbb{R}^n$ then

$$\dim(\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)) = n$$

and so by Exercise 2.138, $\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) = \mathbb{R}^n$ and so $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$. 
Theorem 6.1 (continued 4)

Proof (continued). So each $\vec{b} \in \mathbb{R}^n$ can be written as

$$\vec{b} = \left( r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k \right) + \left( s_{k+1} \vec{v}_{k+1} + s_{k+2} \vec{v}_{k+2} + \cdots + s_n \vec{v}_n \right),$$

where $r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k \in W$ and $s_{k+1} \vec{v}_{k+1} + s_{k+2} \vec{v}_{k+2} + \cdots + s_n \vec{v}_n \in W^\perp$, for unique $r_1, r_2, \ldots, r_k, s_{k+1}, s_{k+2}, \ldots, s_n$ (by Definition 1.17, “Basis for a Subspace”). So any $\vec{b} \in \mathbb{R}^n$ can be expressed in the form $\vec{b} = \vec{b}_W + \vec{b}_{W^\perp}$ where $\vec{b}_W \in W$ and $\vec{b}_{W^\perp} \in W^\perp$. Since each vector in $\mathbb{R}^n$ is a unique linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then the choice of $\vec{b}_W$ and $\vec{b}_{W^\perp}$ are unique.

Page 335 Example 6

Page 335 Example 6. Consider the inner product space $P_{[0,1]}$ of all polynomial functions defined on the interval $[0,1]$ with inner product

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x) \, dx.$$ 

Find the projection of $f(x) = x$ on $\text{sp}(1)$ and then find the projection of $x$ on $\text{sp}(1)^\perp$.

Solution. We follow the definition of the projection of $\vec{b}$ on $\text{sp}(\vec{a})$ in $\mathbb{R}^n$,

$$\vec{b} = \text{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a},$$

but instead of dot products in $\mathbb{R}^n$ we use the inner product in $P_{[0,1]}$. So the desired projection, with $\vec{b} = x$ and $\vec{a} = 1$, is

$$\frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} = \frac{\int_0^1 x \cdot 1 \, dx}{\int_0^1 1 \cdot 1 \, dx} = \frac{(1/2)x^2|_0^1}{x^1|_0^1} = \frac{1}{2}.$$
Let $A$ be an $m \times n$ matrix.

(a) Prove that the set $W$ of row vectors $\bar{x}$ in $\mathbb{R}^m$ such that $\bar{x}A = \bar{0}$ is a subspace of $\mathbb{R}^m$.

(b) Prove that the subspace $W$ in part (a) and the column space of $A$ are orthogonal complements in $\mathbb{R}^m$.

**Proof.** (a) We use Definition 1.16, “Subspace of $\mathbb{R}^n$.” Let $W = \{\bar{x} \in \mathbb{R}^m \mid \bar{x}A = \bar{0}\}$. We must check $W$ for closure under vector addition and scalar multiplication. Let $\bar{x}_1, \bar{x}_2 \in W$ and let $r$ be a scalar.

Then:

$$(\bar{x}_1 + \bar{x}_2)A = \bar{x}_1A + \bar{x}_2A \text{ by Theorem 1.3.A(10),}
= \bar{0} + \bar{0} \text{ since } \bar{x}_1, \bar{x}_2 \in W
= \bar{0}.$$

So both $\bar{x}_1 + \bar{x}_2 \in W$ and $r\bar{x}_1 \in W$. That is, $W$ is closed under vector addition and scalar multiplication. By Definition 1.16, $W$ is a subspace of $\mathbb{R}^m$. □

(b) Let $A$ be an $m \times n$ matrix. Prove that the subspace $W$ in part (a) and the column space of $A$ are orthogonal complements in $\mathbb{R}^m$.

**Proof.** Recall that by Definition 1.8, “Matrix Product,” the $(i,j)$ entry of the matrix product $AB$ is the dot product of the $i$th row of $A$ with the $j$th column of $B$. □

So for $\bar{x} \in W$ (here we treat row vector $\bar{x} \in \mathbb{R}^m$ as a $1 \times m$ matrix) we have that $\bar{x}A$ is a $1 \times n$ matrix (or a row vector in $\mathbb{R}^n$) and for $\bar{x} \in W$ we have $\bar{x}A = \bar{0} \in \mathbb{R}^n$. So the $j$th entry of $\bar{x}A = \bar{0}$ is the dot product of $\bar{x}$ with the $j$th column of $A$ and, since $\bar{x}A = \bar{0}$, this dot product must be 0 for each $j = 1, 2, \ldots, n$. So by Definition 1.7, “Perpendicular or Orthogonal Vectors,” each $\bar{x} \in W$ is orthogonal to each column of $A$. Also, by definition, $W$ contains all vectors $\bar{x}$ in $\mathbb{R}^m$ which satisfy $\bar{x}A = \bar{0}$ (i.e., all vectors $\bar{x}$ in $\mathbb{R}^m$ which are perpendicular to all columns of $A$). The column space of $A$ is the span of the columns of $A$ and since $\bar{x} \in W$ is orthogonal to each column of $A$ then $\bar{x}$ is orthogonal to each vector which is in the span of the columns of $A$. Conversely, any vector $\bar{x}$ in the orthogonal complement of the column space of $A$ must be orthogonal to all linear combinations of the columns of $A$; in particular such $\bar{x}$ must by orthogonal to each column of $A$ and hence such $\bar{x}$ is in $W$. So the orthogonal complement of the column space of $A$ is $W$. □

Let $W$ be a subspace of $\mathbb{R}^n$ with orthogonal complement $W^\perp$. Writing $\bar{a} = \bar{a}_W + \bar{a}_{W^\perp}$, as in Theorem 6.1, prove that $||\bar{a}||^2 = ||\bar{a}_W||^2 + ||\bar{a}_{W^\perp}||^2$.

**Solution.** By Note 1.2.3, $||\bar{a}||^2 = \bar{a} \cdot \bar{a}$, so we have

$$||\bar{a}||^2 = (\bar{a}_W + \bar{a}_{W^\perp}) \cdot (\bar{a}_W + \bar{a}_{W^\perp})$$

$$= \bar{a}_W \cdot \bar{a}_W + \bar{a}_W \cdot \bar{a}_{W^\perp} + \bar{a}_{W^\perp} \cdot \bar{a}_W + \bar{a}_{W^\perp} \cdot \bar{a}_{W^\perp}$$

by Theorem 1.3, “Properties of Dot Products”

$$= ||\bar{a}_W||^2 + \bar{a}_W \cdot \bar{a}_{W^\perp} + \bar{a}_{W^\perp} \cdot \bar{a}_W + ||\bar{a}_{W^\perp}||^2$$

by Note 1.2.A

$$= ||\bar{a}_W||^2 + 0 + 0 + ||\bar{a}_{W^\perp}||^2$$

since $\bar{a}_W$ and $\bar{a}_{W^\perp}$ are orthogonal.

Not taking square roots (and observing that $||\bar{a}||$ is nonnegative) gives $||\bar{a}|| = \sqrt{||\bar{a}_W||^2 + ||\bar{a}_{W^\perp}||^2}$. □