Chapter 6: Orthogonality
Section 6.1. Projections—Proofs of Theorems
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Page 336 number 4

**Page 336 number 4.** Find the projection of $[1, 2, 1]$ on the line with parametric equation $x = 3t$, $y = t$, $z = 2t$ in $\mathbb{R}^3$.

**Solution.** A line is a translation of a one-dimensional subspace and is of the form $\vec{x} = t\vec{d} + \vec{a}$ where $\vec{d}$ is the direction vector and $\vec{a}$ is a translation vector (see Section 2.5, “Lines, Planes, and Other Flats”). Here, $\vec{d} = [3, 1, 2]$ and $\vec{a} = [0, 0, 0]$ so, in fact, the line is not translated and so is a subspace spanned by $\vec{d} = [3, 1, 2]$. 

\[ \begin{align*}
\vec{p} &= \text{proj}_{\vec{d}}(\vec{b}) = \frac{\vec{b} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} \\
&= \frac{[1, 2, 1] \cdot [3, 1, 2]}{[3, 1, 2] \cdot [3, 1, 2]} [3, 1, 2] \\
&= \frac{7}{14} [3, 1, 2] = \frac{3}{2}, \frac{1}{2}, 1.
\end{align*} \]
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\[
\vec{p} = \text{proj}_{\vec{d}}(\vec{b}) = \frac{\vec{b} \cdot \vec{d}}{\vec{d} \cdot \vec{d}} \vec{d} = \frac{[1, 2, 1] \cdot [3, 1, 2]}{[3, 1, 2] \cdot [3, 1, 2]} [3, 1, 2]
\]

\[
= \frac{(1)(3) + (2)(1) + (1)(2)}{3^2 + 1^2 + 2^2} [3, 1, 2] = \frac{7}{14} [3, 1, 2] = \frac{3}{2}, \frac{1}{2}, 1
\]
Page 336 number 4

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\[
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\]

\[
= \frac{(1)(3) + (2)(1) + (1)(2)}{3^2 + 1^2 + 2^2} [3, 1, 2] = \frac{7}{14} [3, 1, 2] = [3/2, 1/2, 1].
\]

□
Page 336 number 10

**Page 336 number 10.** Find the orthogonal complement of the plane $2x + y + 3z = 0$ in $\mathbb{R}^3$.

**Solution.** A plane is a translation of a two-dimensional space of the form
\[
\mathbf{x} = t_1 \mathbf{d}_1 + t_2 \mathbf{d}_2 + \mathbf{a}
\]
where $\mathbf{d}_1$ and $\mathbf{d}_2$ form a basis for the two-dimensional space and $\mathbf{a}$ is a translation vector (see Section 2.5, “Lines, Planes, and Other Flats”). Here, we can take $\mathbf{a} = \mathbf{0}$ so that the plane is not translated and is in fact a subspace of $\mathbb{R}^3$. So we just need a basis for the subspace.
Find the orthogonal complement of the plane $2x + y + 3z = 0$ in $\mathbb{R}^3$.

**Solution.** A plane is a translation of a two-dimensional space of the form $\vec{x} = t_1 \vec{d}_1 + t_2 \vec{d}_2 + \vec{a}$ where $\vec{d}_1$ and $\vec{d}_2$ form a basis for the two-dimensional space and $\vec{a}$ is a translation vector (see Section 2.5, “Lines, Planes, and Other Flats”). Here, we can take $\vec{a} = \vec{0}$ so that the plane is not translated and is in fact a subspace of $\mathbb{R}^3$. So we just need a basis for the subspace. We pick two linearly independent vectors in the subspace, say $\vec{d}_1 = [1, -2, 0]$ and $\vec{d}_2 = [0, -3, 1]$ (though there are infinitely many such choices). Then using the technique described above, we take

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

and find the nullspace of $A$ by considering the system of equations $A\vec{x} = \vec{0}$ (see Note 6.1.A):
Page 336 number 10. Find the orthogonal complement of the plane $2x + y + 3z = 0$ in $\mathbb{R}^3$.

Solution. A plane is a translation of a two-dimensional space of the form $\vec{x} = t_1 \vec{d}_1 + t_2 \vec{d}_2 + \vec{a}$ where $\vec{d}_1$ and $\vec{d}_2$ form a basis for the two-dimensional space and $\vec{a}$ is a translation vector (see Section 2.5, “Lines, Planes, and Other Flats”). Here, we can take $\vec{a} = \vec{0}$ so that the plane is not translated and is in fact a subspace of $\mathbb{R}^3$. So we just need a basis for the subspace. We pick two linearly independent vectors in the subspace, say $\vec{d}_1 = [1, -2, 0]$ and $\vec{d}_2 = [0, -3, 1]$ (though there are infinitely many such choices). Then using the technique described above, we take $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -3 & 1 \end{bmatrix}$ and find the nullspace of $A$ by considering the system of equations $A\vec{x} = \vec{0}$ (see Note 6.1.A):
Solution (continued).

\[
\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & -3 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - (2/3)R_2} \begin{bmatrix} 1 & 0 & -2/3 & 0 \\ 0 & -3 & 1 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -1/3 & 0 \end{bmatrix}.
\]

So we have

\[
x_1 - (2/3)x_3 = 0 \\
x_2 - (1/3)x_3 = 0 \text{ or } x_2 = (1/3)x_3
\]

or with \(x_3 = 3t\) as a free variable, \(x_1 = 2t, x_2 = t,\) and \(x_3 = 3t.\)
Solution (continued).

\[
[A \mid \vec{0}] = \begin{bmatrix}
1 & -2 & 0 & 0 \\
0 & -3 & 1 & 0
\end{bmatrix}
\begin{array}{c}
R_1 \rightarrow R_1 - (2/3)R_2 \\
R_2 \rightarrow R_2 / (-3)
\end{array}
\begin{bmatrix}
1 & 0 & -2/3 & 0 \\
0 & -3 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & -2/3 & 0 \\
0 & 1 & -1/3 & 0
\end{bmatrix}.
\]

So we have

\[
x_1 - (2/3)x_3 = 0 \quad \Rightarrow \quad x_1 = (2/3)x_3
\]
\[
x_2 - (1/3)x_3 = 0 \quad \text{or} \quad x_2 = (1/3)x_3
\]
\[
x_3 = x_3
\]

or with \(x_3 = 3t\) as a free variable, \(x_1 = 2t\), \(x_2 = t\), and \(x_3 = 3t\). So \(W^\perp\) is the nullspace of \(A\): \(W^\perp = \text{sp}([2, 1, 3]). \) □
Solution (continued).

\[
[A \mid \tilde{0}] = \begin{bmatrix}
1 & -2 & 0 & 0 \\
0 & -3 & 1 & 0
\end{bmatrix}
\overset{R_1 \rightarrow R_1 - (2/3)R_2}{\Rightarrow}
\begin{bmatrix}
1 & 0 & -2/3 & 0 \\
0 & -3 & 1 & 0
\end{bmatrix}
\overset{R_2 \rightarrow R_2 / (-3)}{\Rightarrow}
\begin{bmatrix}
1 & 0 & -2/3 & 0 \\
0 & 1 & -1/3 & 0
\end{bmatrix}.
\]

So we have

\[
\begin{align*}
x_1 - (2/3)x_3 &= 0 \\
x_2 - (1/3)x_3 &= 0
\end{align*}
\]

or with \(x_3 = 3t\) as a free variable, \(x_1 = 2t\), \(x_2 = t\), and \(x_3 = 3t\). So \(W^\perp\) is the nullspace of \(A\): 

\[
W^\perp = \text{sp}([2, 1, 3]).
\]
Theorem 6.1

Theorem 6.1. Properties of $W^\perp$.
The orthogonal complement $W^\perp$ of a subspace $W$ of $\mathbb{R}^n$ has the following properties:

1. $W^\perp$ is a subspace of $\mathbb{R}^n$.
2. $\dim(W^\perp) = n - \dim(W)$.
3. $(W^\perp)^\perp = W$.
4. Each vector $\vec{b} \in \mathbb{R}^n$ can be expressed uniquely in the form $\vec{b} = \vec{b}_W + \vec{b}_{W^\perp}$ for $\vec{b}_W \in W$ and $\vec{b}_{W^\perp} \in W^\perp$.

Proof. Let $\dim(W) = k$, and let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a basis for $W$. Let $A$ be the $k \times n$ matrix having $\vec{v}_i$ as its $i$th row vector for $i = 1, 2, \ldots, k$. 

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Property (1) follows from the fact that $W^\perp$ is the nullspace of matrix $A$, by Note 6.1.A, and therefore is a subspace of $\mathbb{R}^n$. 
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Property (1) follows from the fact that $W^\perp$ is the nullspace of matrix $A$, by Note 6.1.A, and therefore is a subspace of $\mathbb{R}^n$. 
Theorem 6.1 (continued 1)

Proof (continued). For Property 2, consider the rank equation of $A$:

$$\text{rank}(A) + \text{nullity}(A) = n.$$ 

Since $\dim(W) = \text{rank}(A)$ and since $W^\perp$ is the nullspace of $A$, then

$$\dim(W^\perp) = n - \dim(W).$$

For Property 3, we have by Property 1 that $W^\perp$ is a subspace of $\mathbb{R}^n$. By Property 2 we have

$$\dim(W^\perp)_{\perp} = n - \dim(W^\perp) = n - (n - k) = k.$$
Theorem 6.1 (continued 1)

Proof (continued). For Property 2, consider the rank equation of $A$:

$$\text{rank}(A) + \text{nullity}(A) = n.$$ 

Since $\dim(W) = \text{rank}(A)$ and since $W^\perp$ is the nullspace of $A$, then

$$\dim(W^\perp) = n - \dim(W).$$

For Property 3, we have by Property 1 that $W^\perp$ is a subspace of $\mathbb{R}^n$. By

$$\dim(W^\perp)^\perp = n - \dim(W^\perp) = n - (n - k) = k.$$ 

Since very vector in $W$ is orthogonal to subspace $W^\perp$ then $W$ is a

subspace of $(W^\perp)^\perp$ ($(W^\perp)^\perp$ is a subspace of $\mathbb{R}^n$ by two applications of Property 1). Since $W$ and $(W^\perp)^\perp$ have the same dimension then by Exercise 2.1.38, $W$ must be equal to $(W^\perp)^\perp.$
Theorem 6.1 (continued 1)

**Proof (continued).** For Property 2, consider the rank equation of $A$:

$$\text{rank}(A) + \text{nullity}(A) = n.$$  

Since $\dim(W) = \text{rank}(A)$ and since $W^\perp$ is the nullspace of $A$, then $\dim(W^\perp) = n - \dim(W)$.

For Property 3, we have by Property 1 that $W^\perp$ is a subspace of $\mathbb{R}^n$. By Property 2 we have

$$\dim((W^\perp)^\perp) = n - \dim(W^\perp) = n - (n - k) = k.$$  

Since every vector in $W$ is orthogonal to subspace $W^\perp$ then $W$ is a subspace of $(W^\perp)^\perp$ ($(W^\perp)^\perp$ is a subspace of $\mathbb{R}^n$ by two applications of Property 1). Since $W$ and $(W^\perp)^\perp$ have the same dimension then by Exercise 2.1.38, $W$ must be equal to $(W^\perp)^\perp$. 
Theorem 6.1 (continued 2)

**Proof (continued).** For Property 4, let \( \{ \vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n \} \) be a basis for \( n - k \) dimensional (by Property 2) subspace \( W^\perp \). We now show that

\[
\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \cup \{ \vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n \} = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}
\]
is a basis for \( \mathbb{R}^n \). Consider the linear combination

\[
r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k + s_{k+1} \vec{v}_{k+1} + s_{k+2} \vec{v}_{k+2} + \cdots + s_n \vec{v}_n = \vec{0}. \tag{*}
\]

This equation implies

\[
r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k = -s_{k+1} \vec{v}_{k+1} - s_{k+2} \vec{v}_{k+2} - \cdots - s_n \vec{v}_n.
\]
Theorem 6.1 (continued 2)

Proof (continued). For Property 4, let \( \{ \vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n \} \) be a basis for \( n - k \) dimensional (by Property 2) subspace \( W^\perp \). We now show that

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\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \cup \{ \vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n \} = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}
\]

is a basis for \( \mathbb{R}^n \). Consider the linear combination

\[
r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k + s_{k+1} \vec{v}_{k+1} + s_{k+2} \vec{v}_{k+2} + \cdots + s_n \vec{v}_n = \vec{0}. \quad (*)
\]

This equation implies

\[
r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k = -s_{k+1} \vec{v}_{k+1} - s_{k+2} \vec{v}_{k+2} - \cdots - s_n \vec{v}_n.
\]
Theorem 6.1 (continued 3)

Proof (continued). Notice that the vector on the left hand side of this equation is in $W$ and the vector on the right hand side is in $W^\perp$. But both sides of the equation represent the same vector (d’uh, it’s an equation!) so both sides of the equation represent a vector in both $W$ and $W^\perp$. So this vector must be orthogonal to itself. The only vector orthogonal to itself is $\vec{0}$ (since $0 = \vec{v} \cdot \vec{v} = \|\vec{v}\|^2$ implies $\vec{v} = \vec{0}$). Since the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent and

$$r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k = \vec{0}$$

then we must have $r_1 = r_2 = \cdots = r_k = 0$. Similarly, $\vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n$ are linearly independent and

$$s_{k+1}\vec{v}_{k+1} + s_{k+2}\vec{v}_{k+2} + \cdots + s_n\vec{v}_n = \vec{0}$$

implies $s_{k+1} = s_{k+2} = \cdots = s_n = 0$. 


Theorem 6.1 (continued 3)

**Proof (continued).** Notice that the vector on the left hand side of this equation is in $W$ and the vector on the right hand side is in $W^\perp$. But both sides of the equation represent the same vector (d’uh, it’s an equation!) so both sides of the equation represent a vector in both $W$ and $W^\perp$. So this vector must be orthogonal to itself. The only vector orthogonal to itself is $\vec{0}$ (since $0 = \vec{v} \cdot \vec{v} = \|\vec{v}\|^2$ implies $\vec{v} = \vec{0}$). Since the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent and

$$r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k = \vec{0}$$

then we must have $r_1 = r_2 = \cdots = r_k = 0$. Similarly, $\vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n$ are linearly independent and

$$s_{k+1}\vec{v}_{k+1} + s_{k+2}\vec{v}_{k+2} + \cdots + s_n\vec{v}_n = \vec{0}$$

implies $s_{k+1} = s_{k+2} = \cdots = s_n = 0$.

From equation $(\ast)$, we see that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a linearly independent set. Since the set contains $n$ linearly independent vectors in $\mathbb{R}^n$ then $\dim(\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)) = n$ and so by Exercise 2.1.38, $\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) = \mathbb{R}^n$ and so $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$. 
Theorem 6.1 (continued 3)

**Proof (continued).** Notice that the vector on the left hand side of this equation is in $W$ and the vector on the right hand side is in $W^\perp$. But both sides of the equation represent the same vector (d’uh, it’s an equation!) so both sides of the equation represent a vector in both $W$ and $W^\perp$. So this vector must be orthogonal to itself. The only vector orthogonal to itself is $\vec{0}$ (since $0 = \vec{v} \cdot \vec{v} = ||\vec{v}||^2$ implies $\vec{v} = \vec{0}$). Since the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent and $r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k = \vec{0}$ then we must have $r_1 = r_2 = \cdots r_k = 0$. Similarly, $\vec{v}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n$ are linearly independent and $s_{k+1} \vec{v}_{k+1} + s_{k+2} \vec{v}_{k+2} + \cdots + s_n \vec{v}_n = \vec{0}$ implies $s_{k+1} = s_{k+2} = \cdots = s_n = 0$. From equation $(*)$, we see that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a linearly independent set. Since the set contains $n$ linearly independent vectors in $\mathbb{R}^n$ then $\dim(\text{sp}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)) = n$ and so by Exercise 2.1.38, $\text{sp}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n) = \mathbb{R}^n$ and so $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$. 

Theorem 6.1 (continued 4)

Proof (continued). So each $\vec{b} \in \mathbb{R}^n$ can be written as

$$\vec{b} = (r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k) + (s_{k+1} \vec{v}_{k+1} + s_{k+2} \vec{v}_{k+2} + \cdots + s_n \vec{v}_n),$$

where $r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k \in W$ and $s_{k+1} \vec{v}_{k+1} + s_{k+2} \vec{v}_{k+2} + \cdots + s_n \vec{v}_n \in W^\perp$, for unique $r_1, r_2, \ldots, r_k, s_{k+1}, s_{k+2}, \ldots, s_n$ (by Definition 1.17, “Basis for a Subspace”). So any $\vec{b} \in \mathbb{R}^n$ can be expressed in the form $\vec{b} = \vec{b}_W + \vec{b}_{W^\perp}$ where $\vec{b}_W \in W$ and $\vec{b}_{W^\perp} \in W^\perp$. Since each vector in $\mathbb{R}^n$ is a unique linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then the choice of $\vec{b}_W$ and $\vec{b}_{W^\perp}$ are unique.
Theorem 6.1 (continued 4)

**Proof (continued).** So each $\vec{b} \in \mathbb{R}^n$ can be written as

$$\vec{b} = (r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k) + (s_{k+1} \vec{v}_{k+1} + s_{k+2} \vec{v}_{k+2} + \cdots + s_n \vec{v}_n),$$

where $r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k \in W$ and $s_{k+1} \vec{v}_{k+1} + s_{k+2} \vec{v}_{k+2} + \cdots + s_n \vec{v}_n \in W^\perp$, for unique $r_1, r_2, \ldots, r_k, s_{k+1}, s_{k+2}, \ldots, s_n$ (by Definition 1.17, “Basis for a Subspace”). So any $\vec{b} \in \mathbb{R}^n$ can be expressed in the form $\vec{b} = \vec{b}_W + \vec{b}_W^\perp$ where $\vec{b}_W \in W$ and $\vec{b}_W^\perp \in W^\perp$. Since each vector in $\mathbb{R}^n$ is a unique linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then the choice of $\vec{b}_W$ and $\vec{b}_W^\perp$ are unique.
Page 336 number 20(b). Find the projection of $\vec{b} = [-2, 1, 3, -5]$ on to the subspace $W = \text{sp}(\hat{e}_1, \hat{e}_4)$ in $\mathbb{R}^4$.

Solution. We are given a basis for $W = \text{sp}(\hat{e}_1, \hat{e}_4)$, namely $\{\hat{e}_1, \hat{e}_4\}$. Certainly a basis for $W^\perp$ is given by $\{\hat{e}_2, \hat{e}_3\}$. 
**Page 336 number 20(b).** Find the projection of \( \vec{b} = [-2, 1, 3, -5] \) on to the subspace \( W = \text{sp}(\hat{e}_1, \hat{e}_4) \) in \( \mathbb{R}^4 \).

**Solution.** We are given a basis for \( W = \text{sp}(\hat{e}_1, \hat{e}_4) \), namely \( \{\hat{e}_1, \hat{e}_4\} \). Certainly a basis for \( W^\perp \) is given by \( \{\hat{e}_2, \hat{e}_3\} \). So we take the ordered basis \( \{\hat{e}_1, \hat{e}_4, \hat{e}_2, \hat{e}_3\} \) of \( \mathbb{R}^4 \) and we have \( \vec{b} = -2\hat{e}_1 - 5\hat{e}_4 + 1\hat{e}_2 + 3\hat{e}_3 \) (and so the coordinate vector \( \vec{r} \) of \( \vec{b} \) relative to the ordered basis \( \{\hat{e}_1, \hat{e}_4, \hat{e}_2, \hat{e}_3\} \) is \( \vec{r} = [-2, -5, 1, 3] \)).
Page 336 number 20(b). Find the projection of $\vec{b} = [-2, 1, 3, -5]$ on to the subspace $W = \text{sp}(\hat{e}_1, \hat{e}_4)$ in $\mathbb{R}^4$.

**Solution.** We are given a basis for $W = \text{sp}(\hat{e}_1, \hat{e}_4)$, namely $\{\hat{e}_1, \hat{e}_4\}$. Certainly a basis for $W^\perp$ is given by $\{\hat{e}_2, \hat{e}_3\}$. So we take the ordered basis $\{\hat{e}_1, \hat{e}_4, \hat{e}_2, \hat{e}_3\}$ of $\mathbb{R}^4$ and we have $\vec{b} = -2\hat{e}_1 - 5\hat{e}_4 + 1\hat{e}_2 + 3\hat{e}_3$ (and so the coordinate vector $\vec{r}$ of $\vec{b}$ relative to the ordered basis $\{\hat{e}_1, \hat{e}_4, \hat{e}_2, \hat{e}_3\}$ is $\vec{r} = [-2, -5, 1, 3]$). Then by the Note 6.1.B, the projection of $\vec{b}$ on to $W$ is

$$\vec{b}_W = \text{proj}_W(\vec{b}) = r_1\hat{e}_1 + r_2\hat{e}_4 = -2[1, 0, 0, 0] - 5[0, 0, 0, 1] = [-2, 0, 0, -5].$$
Page 336 number 20(b). Find the projection of \( \vec{b} = [-2, 1, 3, -5] \) on to the subspace \( W = \text{sp}(\hat{e}_1, \hat{e}_4) \) in \( \mathbb{R}^4 \).

**Solution.** We are given a basis for \( W = \text{sp}(\hat{e}_1, \hat{e}_4) \), namely \( \{\hat{e}_1, \hat{e}_4\} \). Certainly a basis for \( W^\perp \) is given by \( \{\hat{e}_2, \hat{e}_3\} \). So we take the ordered basis \( \{\hat{e}_1, \hat{e}_4, \hat{e}_2, \hat{e}_3\} \) of \( \mathbb{R}^4 \) and we have \( \vec{b} = -2\hat{e}_1 - 5\hat{e}_4 + 1\hat{e}_2 + 3\hat{e}_3 \) (and so the coordinate vector \( \vec{r} \) of \( \vec{b} \) relative to the ordered basis \( \{\hat{e}_1, \hat{e}_4, \hat{e}_2, \hat{e}_3\} \) is \( \vec{r} = [-2, -5, 1, 3] \)). Then by the Note 6.1.B, the projection of \( \vec{b} \) on to \( W \) is

\[
\vec{b}_W = \text{proj}_W(\vec{b}) = r_1 \hat{e}_1 + r_2 \hat{e}_4 = -2[1, 0, 0, 0] - 5[0, 0, 0, 1] = [-2, 0, 0, -5].
\]

\( \square \)
Page 335 Example 6

**Page 335 Example 6.** Consider the inner product space $\mathcal{P}_{[0,1]}$ of all polynomial functions defined on the interval $[0, 1]$ with inner product

$$\langle p(x), q(x) \rangle = \int_{0}^{1} p(x)q(x) \, dx.$$

Find the projection of $f(x) = x$ on $\text{sp}(1)$ and then find the projection of $x$ on $\text{sp}(1)^\perp$.

**Solution.** We follow the definition of the projection $\vec{p}$ of $\vec{b}$ on $\text{sp}(\vec{a})$ in $\mathbb{R}^n$,

$$\vec{p} = \text{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a},$$

but instead of dot products in $\mathbb{R}^n$ we use the inner product in $\mathcal{P}_{[0,1]}$. 

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Page 335 Example 6

Page 335 Example 6. Consider the inner product space $\mathcal{P}_{[0,1]}$ of all polynomial functions defined on the interval $[0, 1]$ with inner product

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x) \, dx.$$ 

Find the projection of $f(x) = x$ on $\text{sp}(1)$ and then find the projection of $x$ on $\text{sp}(1)^\perp$.

Solution. We follow the definition of the projection $\vec{p}$ of $\vec{b}$ on $\text{sp}(\vec{a})$ in $\mathbb{R}^n$, $\vec{p} = \text{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a}$, but instead of dot products in $\mathbb{R}^n$ we use the inner product in $\mathcal{P}_{[0,1]}$. So the desired projection, with $\vec{b} = x$ and $\vec{a} = 1$, is

$$\frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} = \frac{\int_0^1 x \cdot 1 \, dx}{\int_0^1 1 \cdot 1 \, dx} = \frac{(1/2)x^2|_0^1}{x|_0^1} = \frac{1}{2}.$$
Example 6. Consider the inner product space $\mathcal{P}_{[0,1]}$ of all polynomial functions defined on the interval $[0, 1]$ with inner product

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x) \, dx.$$ 

Find the projection of $f(x) = x$ on $\text{sp}(1)$ and then find the projection of $x$ on $\text{sp}(1)^\perp$.

Solution. We follow the definition of the projection $\vec{p}$ of $\vec{b}$ on $\text{sp}(\vec{a})$ in $\mathbb{R}^n$, $\vec{p} = \text{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a}$, but instead of dot products in $\mathbb{R}^n$ we use the inner product in $\mathcal{P}_{[0,1]}$. So the desired projection, with $\vec{b} = x$ and $\vec{a} = 1$, is

$$\frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} = \frac{\int_0^1 x \cdot 1 \, dx}{\int_0^1 1 \cdot 1 \, dx} = \frac{\left(\frac{1}{2}\right)x^2|_0^1}{x|_0^1} = \frac{1}{2}.$$
Page 335 Example 6 (continued)

**Page 335 Example 6.** Consider the inner product space \( \mathcal{P}_{[0,1]} \) of all polynomial functions defined on the interval \([0, 1]\) with inner product

\[
\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x) \, dx.
\]

Find the projection of \( f(x) = x \) on \( \text{sp}(1) \) and then find the projection of \( x \) on \( \text{sp}(1)^\perp \).

**Solution (continued).** Notice that with \( W = \text{sp}(1) \) then we have from Definition 6.2 that \( \vec{b} = \vec{b}_W + \vec{b}_{W^\perp} \) and we can find \( \vec{b}_{W^\perp} \) (where \( W^\perp = \text{sp}(1)^\perp \)) as \( \vec{b}_{W^\perp} = \vec{b} - \vec{b}_W = x - 1/2 \). \( \square \)
Page 337 number 26

Let $A$ be an $m \times n$ matrix.

(a) Prove that the set $W$ of row vectors $\vec{x}$ in $\mathbb{R}^m$ such that $\vec{x}A = \vec{0}$ is a subspace of $\mathbb{R}^m$.

(b) Prove that the subspace $W$ in part (a) and the column space of $A$ are orthogonal complements in $\mathbb{R}^m$.

Proof. (a) We use definition 1.16, “Subspace of $\mathbb{R}^n$.” Let $W = \{\vec{x} \in \mathbb{R}^m \mid \vec{x}A = \vec{0}\}$. We must check $W$ for closure under vector addition and scalar multiplication. Let $\vec{x}_1, \vec{x}_2 \in W$ and let $r$ be a scalar.
Let $A$ be an $m \times n$ matrix.

(a) Prove that the set $W$ of row vectors $\vec{x}$ in $\mathbb{R}^m$ such that $\vec{x}A = \vec{0}$ is a subspace of $\mathbb{R}^m$.

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$$(\vec{x}_1 + \vec{x}_2)A = \vec{x}_1A + \vec{x}_2A \text{ by Theorem 1.3.A(10), “Distribution Laws of Matrix Multiplication”}
$$

(here we treat $\vec{x}$ as a matrix)

$= \vec{0} + \vec{0} \text{ since } \vec{x}_1, \vec{x}_2 \in W$

$= \vec{0}$,
Let $A$ be an $m \times n$ matrix.

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(here we treat $\vec{x}$ as a matrix)

$$= \vec{0} + \vec{0} \text{ since } \vec{x}_1, \vec{x}_2 \in W$$

$$= \vec{0},$$
Proof (continued). . . and

\[(r\vec{x}_1)A = r(\vec{x}_1 A) \text{ by Theorem 1.3.A(7), "Scalars Pull Through"}\]
\[= r\vec{0} \text{ since } \vec{x}_1 \in W\]
\[= \vec{0}.\]

So both \(\vec{x}_1 + \vec{x}_2 \in W\) and \(r\vec{x}_1 \in W\). That is, \(W\) is closed under vector addition and scalar multiplication. By Definition 1.16, \(W\) is a subspace of \(\mathbb{R}^m\). \(\square\)
Proof (continued). ... and

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(b) Let $A$ be an $m \times n$ matrix. Prove that the subspace $W$ in part (a) and the column space of $A$ are orthogonal complements in $\mathbb{R}^m$. 
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(b) Let \(A\) be an \(m \times n\) matrix. Prove that the subspace \(W\) in part (a) and the column space of \(A\) are orthogonal complements in \(\mathbb{R}^m\).

Proof. Recall that by Definition 1.8, “Matrix Product,” the \((i, j)\) entry of the matrix product \(AB\) is the dot product of the \(i\)th row of \(A\) with the \(j\)th column of \(B\).
Proof (continued). ... and

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So both \(\vec{x}_1 + \vec{x}_2 \in W\) and \(r\vec{x}_1 \in W\). That is, \(W\) is closed under vector addition and scalar multiplication. By Definition 1.16, \(W\) is a subspace of \(\mathbb{R}^m\).

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Proof. Recall that by Definition 1.8, “Matrix Product,” the \((i,j)\) entry of the matrix product \(AB\) is the dot product of the \(i\)th row of \(A\) with the \(j\)th column of \(B\).
Proof (continued). So for $\vec{x} \in W$ (here we treat row vector $\vec{x} \in \mathbb{R}^m$ as a $1 \times m$ matrix) we have that $\vec{x}A$ is a $1 \times n$ matrix (or a row vector in $\mathbb{R}^n$) and for $\vec{x} \in W$ we have $\vec{x}A = \vec{0} \in \mathbb{R}^n$. So the $j$th entry of $\vec{x}A = \vec{0}$ is the dot product of $\vec{x}$ with the $j$th column of $A$ and, since $\vec{x}A = \vec{0}$, this dot product must be 0 for each $j = 1, 2, \ldots, n$. So by Definition 1.7, “Perpendicular or Orthogonal Vectors,” each $\vec{x} \in W$ is orthogonal to each column of $A$. Also, by definition, $W$ contain all vectors $\vec{x}$ in $\mathbb{R}^m$ which satisfy $\vec{x}A = \vec{0}$ (i.e., all vectors $\vec{x}$ in $\mathbb{R}^m$ which are perpendicular to all columns of $A$). The column space of $A$ is the span of the columns of $A$ and since $\vec{x} \in W$ is orthogonal to each column of $A$ then $\vec{x}$ is orthogonal to each vector which is in the span of the columns of $A$. 
Proof (continued). So for $\vec{x} \in W$ (here we treat row vector $\vec{x} \in \mathbb{R}^m$ as a $1 \times m$ matrix) we have that $\vec{x}A$ is a $1 \times n$ matrix (or a row vector in $\mathbb{R}^n$) and for $\vec{x} \in W$ we have $\vec{x}A = \vec{0} \in \mathbb{R}^n$. So the $j$th entry of $\vec{x}A = \vec{0}$ is the dot product of $\vec{x}$ with the $j$th column of $A$ and, since $\vec{x}A = \vec{0}$, this dot product must be 0 for each $j = 1, 2, \ldots, n$. So by Definition 1.7, “Perpendicular or Orthogonal Vectors,” each $\vec{x} \in W$ is orthogonal to each column of $A$. Also, by definition, $W$ contain all vectors $\vec{x}$ in $\mathbb{R}^m$ which satisfy $\vec{x}A = \vec{0}$ (i.e., all vectors $\vec{x}$ in $\mathbb{R}^m$ which are perpendicular to all columns of $A$). The column space of $A$ is the span of the columns of $A$ and since $\vec{x} \in W$ is orthogonal to each column of $A$ then $\vec{x}$ is orthogonal to each vector which is in the span of the columns of $A$. Conversely, any vector $\vec{x}$ in the orthogonal complement of the column space of $A$ must be orthogonal to all linear combinations of the columns of $A$; in particular such $\vec{x}$ must by orthogonal to each column of $A$ and hence such $\vec{x}$ is in $W$. So the orthogonal complement of the column space of $A$ is $W$. \qed
Proof (continued). So for $\vec{x} \in W$ (here we treat row vector $\vec{x} \in \mathbb{R}^m$ as a $1 \times m$ matrix) we have that $\vec{x}A$ is a $1 \times n$ matrix (or a row vector in $\mathbb{R}^n$) and for $\vec{x} \in W$ we have $\vec{x}A = \vec{0} \in \mathbb{R}^n$. So the $j$th entry of $\vec{x}A = \vec{0}$ is the dot product of $\vec{x}$ with the $j$th column of $A$ and, since $\vec{x}A = \vec{0}$, this dot product must be 0 for each $j = 1, 2, \ldots, n$. So by Definition 1.7, “Perpendicular or Orthogonal Vectors,” each $\vec{x} \in W$ is orthogonal to each column of $A$. Also, by definition, $W$ contain all vectors $\vec{x}$ in $\mathbb{R}^m$ which satisfy $\vec{x}A = \vec{0}$ (i.e., all vectors $\vec{x}$ in $\mathbb{R}^m$ which are perpendicular to all columns of $A$). The column space of $A$ is the span of the columns of $A$ and since $\vec{x} \in W$ is orthogonal to each column of $A$ then $\vec{x}$ is orthogonal to each vector which is in the span of the columns of $A$. Conversely, any vector $\vec{x}$ in the orthogonal complement of the column space of $A$ must be orthogonal to all linear combinations of the columns of $A$; in particular such $\vec{x}$ must by orthogonal to each column of $A$ and hence such $\vec{x}$ is in $W$. So the orthogonal complement of the column space of $A$ is $W$. \(\square\)
Page 337 number 28. Let $W$ be a subspace of $\mathbb{R}^n$ with orthogonal complement $W^\perp$. Writing $\vec{a} = \vec{a}_W + \vec{a}_{W^\perp}$, as in Theorem 6.1, prove that $\|\vec{a}\| = \sqrt{\|\vec{a}_W\|^2 + \|\vec{a}_{W^\perp}\|^2}$.

Solution. By Note 1.2.A, $\|\vec{a}\|^2 = \vec{a} \cdot \vec{a}$, so we have
Let $W$ be a subspace of $\mathbb{R}^n$ with orthogonal complement $W^\perp$. Writing $\vec{a} = \vec{a}_W + \vec{a}_{W^\perp}$, as in Theorem 6.1, prove that $\|\vec{a}\| = \sqrt{\|\vec{a}_W\|^2 + \|\vec{a}_{W^\perp}\|^2}$.

**Solution.** By Note 1.2.A, $\|\vec{a}\|^2 = \vec{a} \cdot \vec{a}$, so we have

$$\|\vec{a}\|^2 = (\vec{a}_W + \vec{a}_{W^\perp}) \cdot (\vec{a}_W + \vec{a}_{W^\perp})$$

$$= \vec{a}_W \cdot \vec{a}_W + \vec{a}_W \cdot \vec{a}_{W^\perp} + \vec{a}_{W^\perp} \cdot \vec{a}_W + \vec{a}_{W^\perp} \cdot \vec{a}_{W^\perp}$$

by Theorem 1.3, “Properties of Dot Products”

$$= \|\vec{a}_W\|^2 + \vec{a}_W \cdot \vec{a}_{W^\perp} + \vec{a}_{W^\perp} \cdot \vec{a}_W + \|\vec{a}_{W^\perp}\|^2$$ by Note 1.2.A

$$= \|\vec{a}_W\|^2 + 0 + 0 + \|\vec{a}_{W^\perp}\|^2$$ since $\vec{a}_W$ and $\vec{a}_{W^\perp}$ are orthogonal.
Page 337 number 28

Let $W$ be a subspace of $\mathbb{R}^n$ with orthogonal complement $W^\perp$. Writing $\vec{a} = \vec{a}_W + \vec{a}_{W^\perp}$, as in Theorem 6.1, prove that
\[ \|\vec{a}\| = \sqrt{\|\vec{a}_W\|^2 + \|\vec{a}_{W^\perp}\|^2}. \]

Solution. By Note 1.2.A, $\|\vec{a}\|^2 = \vec{a} \cdot \vec{a}$, so we have
\[
\|\vec{a}\|^2 = (\vec{a}_W + \vec{a}_{W^\perp}) \cdot (\vec{a}_W + \vec{a}_{W^\perp}) \\
= \vec{a}_W \cdot \vec{a}_W + \vec{a}_{W^\perp} \cdot \vec{a}_{W^\perp} + \vec{a}_W \cdot \vec{a}_{W^\perp} + \vec{a}_{W^\perp} \cdot \vec{a}_W + \vec{a}_{W^\perp} \cdot \vec{a}_{W^\perp}
\]
by Theorem 1.3, “Properties of Dot Products”
\[
= \|\vec{a}_W\|^2 + \|\vec{a}_{W^\perp}\|^2 + \|\vec{a}_{W^\perp}\|^2 + \|\vec{a}_W\|^2 + \|\vec{a}_{W^\perp}\|^2 \\n= \|\vec{a}_W\|^2 + 0 + 0 + \|\vec{a}_{W^\perp}\|^2 \text{ by Note 1.2.A}
\]
Not taking square roots (and observing that $\|\vec{a}\|$ is nonnegative) gives
\[ \|\vec{a}\| = \sqrt{\|\vec{a}_W\|^2 + \|\vec{a}_{W^\perp}\|^2}. \]

Page 337 number 28. Let $W$ be a subspace of $\mathbb{R}^n$ with orthogonal complement $W^\perp$. Writing $\vec{a} = \vec{a}_W + \vec{a}_{W^\perp}$, as in Theorem 6.1, prove that $\|\vec{a}\| = \sqrt{\|\vec{a}_W\|^2 + \|\vec{a}_{W^\perp}\|^2}$.

Solution. By Note 1.2.A, $\|\vec{a}\|^2 = \vec{a} \cdot \vec{a}$, so we have

\[
\|\vec{a}\|^2 = (\vec{a}_W + \vec{a}_{W^\perp}) \cdot (\vec{a}_W + \vec{a}_{W^\perp}) \\
= \vec{a}_W \cdot \vec{a}_W + \vec{a}_W \cdot \vec{a}_{W^\perp} + \vec{a}_{W^\perp} \cdot \vec{a}_W + \vec{a}_{W^\perp} \cdot \vec{a}_{W^\perp} \\
\text{by Theorem 1.3, “Properties of Dot Products”} \\
= \|\vec{a}_W\|^2 + \vec{a}_W \cdot \vec{a}_{W^\perp} + \vec{a}_{W^\perp} \cdot \vec{a}_W + \|\vec{a}_{W^\perp}\|^2 \text{ by Note 1.2.A} \\
= \|\vec{a}_W\|^2 + 0 + 0 + \|\vec{a}_{W^\perp}\|^2 \text{ since } \vec{a}_W \text{ and } \vec{a}_{W^\perp} \text{ are orthogonal.}
\]

Not taking square roots (and observing that $\|\vec{a}\|$ is nonnegative) gives $\|\vec{a}\| = \sqrt{\|\vec{a}_W\|^2 + \|\vec{a}_{W^\perp}\|^2}$. \qed