Chapter 6: Orthogonality
Section 6.2. The Gram-Schmidt Process—Proofs of Theorems

Theorem 6.2

**Theorem 6.2. Orthogonal Bases.**
Let \( \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \) be an orthogonal set of nonzero vectors in \( \mathbb{R}^n \). Then this set is independent and consequently is a basis for the subspace \( \text{sp}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) \).

**Proof.** Let \( j \) be an integer between \( 2 \) and \( k \). Consider

\[
\vec{v}_j = s_1 \vec{v}_1 + s_2 \vec{v}_2 + \cdots + s_{j-1} \vec{v}_{j-1}.
\]

If we take the dot product of each side of this equation with \( \vec{v}_j \) then, since the set of vectors is orthogonal, we get \( \vec{v}_j \cdot \vec{v}_j = 0 \), which contradicts the hypothesis that \( \vec{v}_j \neq \vec{0} \). Therefore no \( \vec{v}_j \) is a linear combination of its predecessors and by Page 203 Number 37, the set is independent. Therefore the set is a basis for its span. \( \Box \)

Theorem 6.3

**Theorem 6.3. Projection Using an Orthogonal Basis.**
Let \( \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \) be an orthogonal basis for a subspace \( W \) of \( \mathbb{R}^n \), and let \( \vec{b} \in \mathbb{R}^n \). The projection of \( \vec{b} \) on \( W \) is

\[
\vec{b}_W = \text{proj}_W(\vec{b}) = \frac{\vec{b} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \frac{\vec{b} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 + \cdots + \frac{\vec{b} \cdot \vec{v}_k}{\vec{v}_k \cdot \vec{v}_k} \vec{v}_k.
\]

**Proof.** We know from Theorem 6.1 that \( \vec{b} = \vec{b}_W + \vec{b}_{W^\perp} \) where \( \vec{b}_W \) is the projection of \( \vec{b} \) on \( W \) and \( \vec{b}_{W^\perp} \) is the projection of \( \vec{b} \) on \( W^\perp \). Since \( \vec{b}_W \in W \) and \( \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \) is a basis of \( W \), then

\[
\vec{b}_W = r_1 \vec{v}_1 + r_2 \vec{v}_2 + \cdots + r_k \vec{v}_k
\]

for some scalars \( r_1, r_2, \ldots, r_k \). We now find these \( r_i \)'s.

Theorem 6.3 (continued)

**Proof (continued).** Taking the dot product of \( \vec{b} \) with \( \vec{v}_i \) we have

\[
\begin{align*}
\vec{b} \cdot \vec{v}_i &= (\vec{b}_W + \vec{b}_{W^\perp}) \cdot \vec{v}_i = (\vec{b}_W \cdot \vec{v}_i) + (\vec{b}_{W^\perp} \cdot \vec{v}_i) \\
&= (r_1 \vec{v}_1 \cdot \vec{v}_i + r_2 \vec{v}_2 \cdot \vec{v}_i + \cdots + r_k \vec{v}_k \cdot \vec{v}_i) + 0 \\
&= r_i \vec{v}_i \cdot \vec{v}_i.
\end{align*}
\]

Therefore \( r_i = (\vec{b} \cdot \vec{v}_i)/(\vec{v}_i \cdot \vec{v}_i) \) and so

\[
r_i \vec{v}_i = \frac{\vec{b} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i} \vec{v}_i.
\]

Substituting these values of the \( r_i \)'s into the expression for \( \vec{b}_W \) yields the theorem. \( \Box \)
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Consider\
\[
W = \text{sp}([1, -1, 1, 1], [-1, 1, 1, 1], [1, 1, -1, 1]).
\]
Verify that the generating set of $W$ is orthogonal and find the projection of $b = [1, 4, 1, 2]$ on $W$.

**Solution.** We check pairwise for orthogonality of the three generating vectors:

\[
[1, -1, 1, 1] : [1, -1, 1, 1] = (1)(-1) + (-1)(1) + (1)(1) + (1)(1) = -1 - 1 + 1 + 1 = 0,
\]

\[
[1, -1, 1, 1] : [1, 1, -1, 1] = (1)(1) + (-1)(1) + (1)(-1) + (1)(1) = 1 - 1 - 1 + 1 = 0,
\]

\[
[-1, 1, 1, 1] : [1, 1, -1, 1] = (-1)(1) + (1)(1) + (1)(-1) + (1)(1) = -1 + 1 - 1 + 1 = 0.
\]

Since each dot product is 0 then the vectors form an orthogonal set (in fact, an orthogonal basis for $W$, by Theorem 6.2, “Orthogonal Bases”).

**Solution (continued).** By Theorem 6.3, “Projection Using an Orthogonal Basis,” we have the projection of $b$ on $W$ is

\[
\tilde{b}_W = \text{proj}_W(b) = \frac{\tilde{b} \cdot \tilde{v}_1}{\tilde{v}_1 \cdot \tilde{v}_1} \tilde{v}_1 + \frac{\tilde{b} \cdot \tilde{v}_2}{\tilde{v}_2 \cdot \tilde{v}_2} \tilde{v}_2 + \frac{\tilde{b} \cdot \tilde{v}_3}{\tilde{v}_3 \cdot \tilde{v}_3} \tilde{v}_3
\]

where $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ are the three orthogonal generating vectors, so

\[
\tilde{b}_W = \frac{1}{4} \left[ [1, 4, 1, 2] \cdot [1, -1, 1, 1] \right] [1, -1, 1, 1] + \frac{1}{4} \left[ [1, 4, 1, 2] \cdot [-1, 1, 1, 1] \right] [-1, 1, 1, 1] + \frac{1}{4} \left[ [1, 4, 1, 2] \cdot [1, 1, -1, 1] \right] [1, 1, -1, 1] \\
= \frac{0}{4} [1, -1, 1, 1] + \frac{6}{4} [-1, 1, 1, 1] + \frac{6}{4} [1, 1, -1, 1] \\
= 0 [1, -1, 1, 1] + (3/2) [-1, 1, 1, 1] + (3/2) [1, 1, -1, 1] = [0, 3, 0, 3]. \quad \square
\]

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**Theorem 6.4.** Orthonormal Basis (Gram-Schmidt) Theorem.

Let $W$ be a subspace of $\mathbb{R}^n$, let $\{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k\}$ be any basis for $W$, and let

\[
W_j = \text{sp}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_j)
\]

for $j = 1, 2, \ldots, k$.

Then there is an orthonormal basis $\{q_1, q_2, \ldots, q_k\}$ for $W$ such $W_j = \text{sp}(q_1, q_2, \ldots, q_j)$.

**Proof.** We give a recursive construction which will reveal how to apply the Gram-Schmidt Process.

First, let $\tilde{v}_1 = \tilde{a}_1$ (we will create an orthogonal basis $\{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_j\}$ and then normalize each $\tilde{v}_i$ to create an orthonormal set). For $j = 2, 3, \ldots, k$, let $\tilde{p}_j$ be the projection $\tilde{a}_j$ on $W_{j-1} = \text{sp}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_{j-1})$ and let $\tilde{v}_j = \tilde{a}_j - \tilde{p}_j$. This computation of $\tilde{v}_j$ is given symbolically in Figure 6.7.

**Proof (continued).**

Since $\tilde{p}_j$ is the projection of $\tilde{a}_j$ on $W_{j-1}$ then by Theorem 6.1(4), “Properties of $W_{j-1}$,” and Definition 6.2, “Projection of $\tilde{b}$ on $W_j$,” we have

\[
\tilde{a}_j = (\tilde{a}_j)_{W_{j-1}} + (\tilde{a}_j)_{W_{j-1}^\perp} = \tilde{p}_j + (\tilde{a}_j - \tilde{p}_j) = \tilde{p}_j + \tilde{v}_j
\]

(and by Theorem 6.1(4), the choice of $\tilde{p}_j$ and $\tilde{v}_j$ are unique). Since $\tilde{v}_j \in W_{j-1}^\perp$ then $\tilde{v}_j$ is perpendicular to each $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{j-1} \in W_{j-1}$.
Theorem 6.4. Orthonormal Basis (Gram-Schmidt) Theorem.

Let $W$ be a subspace of $\mathbb{R}^n$, let $\{ \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k \}$ be any basis for $W$, and let

$$W_j = \text{sp}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_j)$$

for $j = 1, 2, \ldots, k$.

Then there is an orthonormal basis $\{ \tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_k \}$ for $W$ such that $W_j = \text{sp}(\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_j)$.

Proof (continued). So each set $\{ \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_j \}$ is an orthogonal set of vectors for each $j = 1, 2, \ldots, k$ and since $\{ \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_j \} \subset W_j$ (where $\dim(W_j) = j$) then by Theorem 6.2, "Orthogonal Bases," $\{ \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_j \}$ is a basis for $W_j$.

Finally, define $\tilde{q}_i = \tilde{v}_i / \| \tilde{v}_i \|$ for $i = 1, 2, \ldots, j$. Then $W = \text{sp}(\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_j)$, $\{ \tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_j \}$ is an orthonormal basis for $W_j$, and in particular $\{ \tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_k \}$ is an orthonormal basis for $W$, as claimed.

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Page 348 Number 10 (continued 1)

Solution (continued). . .

$$= [0, 1, 1] - \frac{2}{3} [1, 1, 1] - \frac{1}{6} [1, -2, 1] = \left[ -\frac{2}{3} + \frac{1}{6}, 1 - \frac{2}{3} - \frac{1}{3}, 1 - \frac{2}{3} + \frac{1}{6} \right]$$

$$= \left[ -\frac{1}{2}, 0, \frac{1}{2} \right] = \frac{1}{2} [-1, 0, 1].$$

Finally we normalize $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ to get

$$\tilde{q}_1 = \frac{\tilde{v}_1}{\| \tilde{v}_1 \|} = \frac{[1, 1, 1]}{\| [1, 1, 1] \|} = \left[ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right],$$

$$\tilde{q}_2 = \frac{\tilde{v}_2}{\| \tilde{v}_2 \|} = \frac{\frac{1}{3} [1, -2, 1]}{\| \frac{1}{3} [1, -2, 1] \|} = \left[ \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right],$$

$$\tilde{q}_3 = \frac{\tilde{v}_3}{\| \tilde{v}_3 \|} = \frac{\frac{1}{3} [-1, 0, 1]}{\| \frac{1}{3} [-1, 0, 1] \|} = \left[ \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right].$$

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Page 348 Number 10 (continued 1)

Solution (continued). . .

$$= [0, 1, 1] - \frac{2}{3} [1, 1, 1] - \frac{1}{6} [1, -2, 1] = \left[ -\frac{2}{3} + \frac{1}{6}, 1 - \frac{2}{3} - \frac{1}{3}, 1 - \frac{2}{3} + \frac{1}{6} \right]$$

$$= \left[ -\frac{1}{2}, 0, \frac{1}{2} \right] = \frac{1}{2} [-1, 0, 1].$$

Finally we normalize $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ to get

$$\tilde{q}_1 = \frac{\tilde{v}_1}{\| \tilde{v}_1 \|} = \frac{[1, 1, 1]}{\| [1, 1, 1] \|} = \left[ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right],$$

$$\tilde{q}_2 = \frac{\tilde{v}_2}{\| \tilde{v}_2 \|} = \frac{\frac{1}{3} [1, -2, 1]}{\| \frac{1}{3} [1, -2, 1] \|} = \left[ \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right],$$

$$\tilde{q}_3 = \frac{\tilde{v}_3}{\| \tilde{v}_3 \|} = \frac{\frac{1}{3} [-1, 0, 1]}{\| \frac{1}{3} [-1, 0, 1] \|} = \left[ \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right].$$

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Solution (continued). So an orthonormal basis is

$$\{ \tilde{q}_1, \tilde{q}_2, \tilde{q}_3 \} = \left\{ \left[ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right], \left[ \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right], \left[ -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right] \right\}.$$
Corollary 1. \textbf{QR-Factorization.}

Let \( A \) be an \( n \times k \) matrix with independent column vectors in \( \mathbb{R}^n \). There exists an \( n \times k \) matrix \( Q \) with orthonormal column vectors and an upper-triangular invertible \( k \times k \) matrix \( R \) such that \( A = QR \).

\textbf{Proof.} Denote the columns of \( A \) as \( \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k \). In the proof of Theorem 6.4 we saw that there exists \( \{ \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_j \} \) and \( \{ \tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_j \} \) both bases of \( W_j = \text{sp}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_j) \). So each \( \tilde{a}_j \) is a unique linear combination of \( \tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_j \):

\[
\tilde{a}_j = r_{1j} \tilde{q}_1 + r_{2j} \tilde{q}_2 + \cdots + r_{jj} \tilde{q}_j \quad \text{for } j = 1, 2, \ldots, k.
\]

Define \( n \times k \) matrix \( A \) with columns \( \tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_k \) and define \( k \times k \) matrix \( R = [r_{ij}] \) where the \( r_{ij} \) are the coefficients given above.

Corollary 1 (continued 2)

More generally, we can state the following:

\textbf{Corollary 1. QR-Factorization.}

Let \( A \) be an \( n \times k \) matrix with independent column vectors in \( \mathbb{R}^n \). There exists an \( n \times k \) matrix \( Q \) with orthonormal column vectors and an upper-triangular invertible \( k \times k \) matrix \( R \) such that \( A = QR \).

\textbf{Proof (continued).} Now if we let the \( i \)th column of \( R \) be vector \( \tilde{r}_i \) then \( QR_i \) is a linear combination of \( \tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_k \) with coefficients \( r_{1i}, r_{2i}, \ldots, r_{ii} \) (see Note 1.3.A) as

\[
QR_i = r_{1i} \tilde{q}_1 + r_{2i} \tilde{q}_2 + \cdots + r_{ii} \tilde{q}_i \quad \text{for } i = 1, 2, \ldots, k.
\]

That is, the \( i \)th column of \( QR \) is \( \tilde{a}_i \) and this holds for \( i = 1, 2, \ldots, k \). So \( A = QR \), as claimed.

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Page 348 Number 26. Find a QR-factorization of \( A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \).

\textbf{Solution.} As seen in the proof of Corollary 1, we need to convert the columns of \( A, \tilde{a}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \) and \( \tilde{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) into an orthonormal basis \( \{ \tilde{q}_1, \tilde{q}_2 \} \) for \( \text{sp}(\tilde{a}_1, \tilde{a}_2) \). We take \( \tilde{v}_1 = \tilde{a}_1 = [0, 1, 0]^T \) and

\[
\tilde{v}_2 = \tilde{a}_2 - \frac{\tilde{a}_1 \cdot \tilde{v}_1}{\tilde{v}_1 \cdot \tilde{v}_1} \tilde{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{[1,1,1]^T \cdot [0,1,0]^T}{[0,1,0]^T \cdot [0,1,0]^T} [0,1,0]^T
\]

\[
= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.
\]
**Corollary 2.** Expansion of an Orthogonal Set to an Orthogonal Basis.
Every orthogonal set of vectors in a subspace $W$ of $\mathbb{R}^n$ can be expanded if necessary to an orthogonal basis of $W$.

**Proof.** An orthogonal set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ of vectors in $W$ is an independent set by Theorem 6.2, and can be expanded to a basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r, \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_s\}$ of $W$ by Theorem 2.3. We apply the Gram-Schmidt Process (Theorem 6.4) to this basis for $W$. Because the $\vec{v}_j$ are already mutually perpendicular, none of them will be changed by the Gram-Schmidt Process (since they are taken first), and so the process yields an orthogonal basis containing the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$.

\[ \square \]
Solution (continued).

\[ \vec{v}_3 = \frac{\vec{a}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{a}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2 \]

\[ = \begin{bmatrix} 0, 1, 0 \end{bmatrix} - \begin{bmatrix} 0, 1, 0 \end{bmatrix} \cdot \frac{1}{\frac{1}{\sqrt{3}}[1, 1, 1]} \cdot \begin{bmatrix} 1, 1, 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2, -1, -1 \end{bmatrix} \cdot \begin{bmatrix} 0, 1, 0 \end{bmatrix} \cdot \frac{1}{\frac{1}{\sqrt{3}}[1, 1, 1]} \cdot \begin{bmatrix} 1, 1, 1 \end{bmatrix} \]

\[ = \begin{bmatrix} 0, 1, 0 \end{bmatrix} - \begin{bmatrix} 0, 1, 0 \end{bmatrix} \cdot \frac{1}{\frac{1}{\sqrt{3}}[1, 1, 1]} \cdot \begin{bmatrix} 1, 1, 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2, -1, -1 \end{bmatrix} \cdot \begin{bmatrix} 0, 1, 0 \end{bmatrix} \cdot \frac{1}{\frac{1}{\sqrt{3}}[1, 1, 1]} \cdot \begin{bmatrix} 1, 1, 1 \end{bmatrix} \]

\[ = \begin{bmatrix} 0 - \frac{1}{3} + \frac{2}{6}, 1 - \frac{1}{3} - \frac{1}{6}, 0 - \frac{1}{3} - \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0, \frac{1}{2}, \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0, 1, -1 \end{bmatrix}. \]

So an orthogonal basis for \( \mathbb{R}^3 \) is \( \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \} \).

Solution (continued). We normalize these vectors to get an orthonormal basis \( \{ \vec{q}_1, \vec{q}_2, \vec{q}_3 \} \) (notice that \( \| \vec{v}_1 \| = 1 \), so we take \( \vec{q}_1 = \vec{v}_1 \)). So

\[ \vec{q}_2 = \frac{\vec{v}_2}{\| \vec{v}_2 \|} = \frac{\frac{1}{3}[2, -1, -1]}{\frac{1}{3} \sqrt{6}} = \frac{1}{\sqrt{6}}[2, -1, -1], \]

and

\[ \vec{q}_3 = \frac{\vec{v}_3}{\| \vec{v}_2 \|} = \frac{\frac{1}{3}[0, 1, -1]}{\frac{1}{3} \sqrt{2}} = \frac{1}{\sqrt{2}}[0, 1, -1]. \]

So an orthonormal basis of \( \mathbb{R}^3 \) including \( \vec{a}_1 = \vec{v}_1 = \vec{q}_1 = \frac{1}{\sqrt{3}}[1, 1, 1] \) is

\[ \left\{ \frac{1}{\sqrt{3}}[1, 1, 1], \frac{1}{\sqrt{6}}[2, -1, -1], \frac{1}{\sqrt{2}}[0, 1, -1] \right\}. \]

Page 348 Number 20. Find an orthonormal basis for \( \mathbb{R}^3 \) that contains the vector \( (1/\sqrt{3})[1, 1, 1] \).

Solution (continued). Notice that this answer depends on the fact that we chose as a spanning set of \( \mathbb{R}^3 \) the given vector along with the standard basis \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \) of \( \mathbb{R}^3 \) (in this order). We could have chosen a different basis or the standard basis but in a different order and we would have gotten a different answer. There are an infinite number of correct answers. □

Page 349 Number 30. Let \( A \) be an \( n \times n \) matrix. Prove that \( A \) has an orthonormal column vector if and only if \( A \) is invertible with inverse \( A^{-1} = A^T \). HINT: Use Exercise 6.3.29 which states: "Let \( A \) be an \( n \times k \) matrix. Prove that the column vectors of \( A \) are orthonormal if and only if \( A^T A = I. \)" NOTE: Exercise 6.3.29 is the inspiration for the definition of "orthogonal matrix" in the next section.

Solution. By Exercise 6.3.29 (with \( k = n \)) we have that the column vectors of \( A \) are orthonormal if and only if \( A^T A = I \). Notice that, since \( A \) and \( A^T \) are \( n \times n \) matrices, by Theorem 1.11, "A Commutivity Property," \( A^T A = I \) implies \( A A^T = I \). So if the column vectors of \( A \) are orthonormal then, by Exercise 6.3.29, \( A^T A = I = A A^T \) and so \( A \) is invertible with \( A^{-1} = A^T \). Conversely, suppose \( A \) is invertible and \( A^{-1} = A^T \). Then

\( A^{-1} A = A^T A = I \) and so by Exercise 6.3.29 the column vectors of \( A \) are orthonormal. □
Page 349 Number 32. Let $V$ be an inner-product space of dimension $n$ and let $B$ be an ordered orthonormal basis for $V$. Prove that, for any vectors $\vec{a}, \vec{c} \in V$, the inner product $\langle \vec{a}, \vec{c} \rangle$ is equal to dot product of the coordinate vectors of $\vec{a}$ and $\vec{c}$ relative to $B$. NOTE: We already know that any two $n$-dimensional vector spaces are isomorphic by the “Fundamental Theorem of Finite Dimensional Vector Spaces,” Theorem 3.3.A, and the isomorphism involves mapping each vector of a given $n$-dimensional vector space to its coordinate vector in $\mathbb{R}^n$. This exercise shows that the inner product structures is also preserved under the same isomorphism so that we can conclude that any two $n$-dimensional inner product spaces are isomorphic (and so any $n$-dimensional inner product space is isomorphic to $\mathbb{R}$ where the inner product on $\mathbb{R}^n$ is the usual dot product).

Proof (continued).

\[
\langle \vec{a}, \vec{c} \rangle = a_1 \vec{b}_1 + a_2 \vec{b}_2 + \cdots + a_n \vec{b}_n, c_1 \vec{b}_1 + c_2 \vec{b}_2 + \cdots + c_n \vec{b}_n
\]

\[= \langle a_1 \vec{b}_1, c_1 \vec{b}_1 \rangle + \langle a_2 \vec{b}_2, c_2 \vec{b}_2 \rangle + \cdots + \langle a_n \vec{b}_n, c_n \vec{b}_n \rangle
\]

\[= \langle \vec{a}_B, \vec{c}_B \rangle = \vec{a} \cdot \vec{c}_B,
\]

where $\vec{a}_B = [a_1, a_2, \ldots, a_n]$ and $\vec{c}_B = [c_1, c_2, \ldots, c_n]$.

Page 349 Number 34. Find an orthonormal basis for $sp(1, x, x^2)$ for $-1 \leq x \leq 1$ if the inner product is defined by $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx$.

Solution. We apply the Gram-Schmidt Process to \{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \{1, x, x^2\}. We simply replace the dot product of $\mathbb{R}^n$ with the inner product given here. Let $\vec{v}_1 = \vec{a}_1 = 1$. Then

\[\vec{v}_2 = \frac{\vec{a}_2 - \frac{\langle \vec{a}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1}{\langle \vec{v}_1, \vec{v}_1 \rangle} = x - \left( \frac{\int_{-1}^{1} x \cdot 1 \, dx}{\int_{-1}^{1} 1 \, dx} \right) 1
\]

\[= x - \left( \frac{\frac{1}{2}(1^2 - (-1)^2)}{x|_{-1}^{1}} \right) \frac{1}{1 - (-1)} = x - \frac{1}{2}(1^2 - 1^2) \frac{1}{2} = x - 0 = x,
\]

and

\[\vec{v}_3 = \frac{\vec{a}_3 - \frac{\langle \vec{a}_3, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 - \frac{\langle \vec{a}_3, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2}{\langle \vec{v}_2, \vec{v}_2 \rangle} = \frac{1}{\sqrt{2}}(1, -1),
\]

\[
\frac{\langle \vec{a}_3, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} = \frac{\frac{1}{2}(1^2 - (-1)^2)}{1} = \frac{1}{2}(1^2 - 1^2) = 0,
\]

\[
\frac{\langle \vec{a}_3, \vec{v}_2 \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} = \frac{\frac{1}{2}(1^2 - (-1)^2)}{1} = \frac{1}{2}(1^2 - 1^2) = 0.
\]
Solution (continued). . .

\[ \tilde{v}_3 = x^2 - \left( \int_{-1}^{1} x^2 \cdot 1 \, dx \right) \left[ \int_{-1}^{1} x \cdot x \, dx \right] \frac{x}{\sqrt{3(1) - (1)}} \]

\[ = x^2 - \left( \frac{1}{3} x^3 \right) \left[ \frac{x}{1} \right] \frac{x}{\sqrt{3(1) - (1)}} \]

\[ = x^3 - \left( \frac{1}{3} \right) x = x^2 - \frac{1}{3}. \]

Finally, we normalize:

\[ \tilde{q}_1 = \frac{\tilde{v}_1}{\| \tilde{v}_1 \|} = \frac{1}{\sqrt{1}} = \frac{1}{\sqrt{\int_{-1}^{1} x^2 \, dx}} = \frac{1}{\sqrt{(1) - (-1)}} = \frac{1}{\sqrt{2}}. \]

\[ \tilde{q}_2 = \frac{\tilde{v}_2}{\| \tilde{v}_2 \|} = \frac{1}{\sqrt{\langle x, x \rangle}} = \frac{1}{\sqrt{\int_{-1}^{1} x^2 \, dx}} = \frac{x}{\sqrt{\frac{3}{2} x^3}} = \cdots \]

\[ \tilde{q}_2 = \frac{x}{\sqrt{\frac{3}{2} (1) - \frac{3}{2} (-1)}} = \frac{x}{\sqrt{\frac{3}{2}}} = \frac{\sqrt{3} x}{\sqrt{2}}. \]

and

\[ \tilde{q}_3 = \frac{\tilde{v}_3}{\| \tilde{v}_3 \|} = \frac{x^2 - \frac{1}{3}}{\sqrt{x^2 - \frac{1}{3}}} = \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1} (x^2 - 1/3)^2 \, dx}} \]

\[ \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1} x^4 - \frac{2}{3} x^2 - \frac{1}{3} \, dx}} = \frac{x^2 - \frac{1}{3}}{\sqrt{\left( \frac{1}{2} x^5 - \frac{2}{3} x^3 + \frac{1}{3} x \right)_{-1}}} = \frac{x^2 - \frac{1}{3}}{\sqrt{8/45}} = \frac{\sqrt{45}}{8} \left( x^2 - \frac{1}{3} \right) \frac{3 \sqrt{5}}{2 \sqrt{2}} \left( x^2 - \frac{1}{3} \right). \]

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Page 349 Number 34. Find an orthonormal basis for \( \text{sp}(1, x, x^2) \) for \(-1 \leq x \leq 1\) if the inner product is defined by \( \langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx \).

Solution (continued). So an orthonormal basis for \( \text{sp}(1, x, x^2) \) is

\[ \left\{ \frac{1}{\sqrt{2}}, \frac{\sqrt{3} x}{\sqrt{2}}, \frac{3 \sqrt{5}}{2 \sqrt{2}} \left( x^2 - \frac{1}{3} \right) \right\}. \]

\[ \square \]