**Linear Algebra**

**Chapter 7. Change of Basis**

Section 7.2. Matrix Representations and Similarity—Proofs of Theorems

---

**Page 406 number 2.** Consider the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^2$ defined by $T([x, y]) = [2x + 3y, x + 2y]$ and ordered bases $B = ([1, -1], [1, 1])$ and $B' = ([2, 3], [1, 2])$. Find the matrix representations of $T$, $R_B$ and $R_{B'}$. Find an invertible matrix $C$ such that $R_{B'} = C^{-1}R_BC$.

**Solution.** We need (omitting some computations):

- $T([1, -1])_B = [0, -1]$  
- $T([1, 1])_B = [1, 4]$  
- $T([2, 3])_{B'} = [18, -23]$  
- $T([1, 2])_{B'} = [-11, -14]$  

So $R_B = [0, 1; -1, 4]$ and ...

---

**Page 406 number 2 (continued 1)**

**Solution (continued).** ...

$$R_{B'} = [T(\mathbf{b}_1)_{B'}, T(\mathbf{b}_2)_{B'}] = \begin{bmatrix} 18 & 11 \\ -23 & -14 \end{bmatrix}.$$  

Notice that we could have formed $R_B$ by direct row reduction of $[\mathbf{b}_1 | T(\mathbf{b}_1) T(\mathbf{b}_2)] \sim [I | R_B]$ (and similarly for $R_{B'}$); see Example 7.2.2 on pages 299 and 400. By Theorem 7.1, $C = C_{B',B}$ so we consider the augmented matrix:

$$[M_B | M_{B'}] = [\mathbf{b}_2 | \mathbf{b}_1 \mathbf{b}_2] = \begin{bmatrix} 1 & 1 & 2 & 1 \\ -1 & 1 & 3 & 2 \end{bmatrix} \overset{R_2 \rightarrow R_2 / 2}{\rightarrow} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 5/2 & 3/2 \end{bmatrix} \overset{R_1 \rightarrow R_1 - R_2}{\rightarrow} \begin{bmatrix} 1 & 0 & -1/2 & -1/2 \\ 0 & 1 & 5/2 & 3/2 \end{bmatrix} = [I | C_{B',B}]$$

So $C = C_{B',B} = \frac{1}{2} \begin{bmatrix} -1/2 & -1/2 \\ 5/2 & 3/2 \end{bmatrix}$. □

---

**Theorem 7.1.** Significance of the Similarity Relationship for Matrices.

Two $n \times n$ matrices are similar if and only if they are matrix representations of the same linear transformation $T$ relative to suitable ordered bases.

**Proof.** Theorem 7.1 shows that matrix representations of the same transformation relative to different bases are similar. Now for the converse. Let $A$ be an $n \times n$ matrix representing transformation $T$, and let $F$ be similar to $A$, say $F = C^{-1}AC$. Since $C$ is invertible, its columns are independent and form a basis for $\mathbb{R}^n$. Let $B$ be the ordered basis having as $j$th vector the $j$th column vector of $C$. Then $C$ is the change-of-coordinates matrix from $B$ to the standard ordered basis $E$. That is, $C = C_{B,E}$. Therefore $F = C^{-1}AC = C_{E,B}A C_{B,E}$ is the matrix representation of $T$ relative to basis $B$. □
**Theorem 7.2. Eigenvalues and Eigenvectors of Similar Matrices.**

Let \( A \) and \( R \) be similar \( n \times n \) matrices, so that \( R = C^{-1}AC \) for some invertible \( n \times n \) matrix \( C \). Let the eigenvalues of \( A \) be the (not necessarily distinct) numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \).

1. The eigenvalues of \( R \) are also \( \lambda_1, \lambda_2, \ldots, \lambda_n \).

**Proof.** The characteristic equation for matrix \( R \) is \( \det(R - \lambda I) \) and so

\[
\det(R - \lambda I) = \det(C^{-1}AC - \lambda I) = \det(C^{-1}(A - \lambda I)C) = \det(C^{-1}) \det(A - \lambda I) \det(C) \text{ by Theorem 4.4}
\]

\[
= \frac{1}{\det(C)} \det(A - \lambda I) \det(C) \text{ by Page 262 number 31}
\]

\[
= \det(A - \lambda I).
\]

Therefore the characteristic equation of \( R \) and \( A \) are the same, and so \( R \) and \( A \) have the same eigenvalues.

---

**Page 407 Number 20.**

Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be the linear transformation defined by \( T([x_1, x_2, x_3]) = [x_1, 4x_2 + 7x_3, 2x_2 - x_3] \). Determine whether \( T \) is diagonalizable.

**Solution.** First we need the standard matrix representation of \( T \):

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 7 & -1 \end{bmatrix}.
\]

For the eigenvalues of \( T \), we calculate the eigenvalues of \( A \) (see Definition 5.2):

\[
\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & 4 - \lambda & 2 \\ 0 & 7 & -1 - \lambda \end{vmatrix} = (1-\lambda)((4-\lambda)(-1-\lambda)-(2)(7))
\]

\[
= (1-\lambda)(-4-3\lambda+\lambda^2-14) = (1-\lambda)(\lambda^2-3\lambda-18) = (1-\lambda)(\lambda-6)(\lambda+3).
\]

---

**Page 407 Number 20 (continued).**

Setting \( \det(A - \lambda I) = (1-\lambda)(\lambda-6)(\lambda+3) = 0 \) we see that the eigenvalues of \( T \) (and \( A \)) are \( \lambda_1 = -3, \lambda_2 = 1, \lambda_3 = 6 \). Since \( A \) is \( 3 \times 3 \) and \( A \) has 3 distinct eigenvalues, then \( A \) is a diagonalizable matrix by Theorem 5.3. So by Theorem 5.5, the algebraic multiplicity of each eigenvalue of \( A \) (and hence of \( T \)) is equal to its geometric multiplicity. So by Definition 7.2, \( \boxed{\text{YES}} \) \( T \) is diagonalizable.