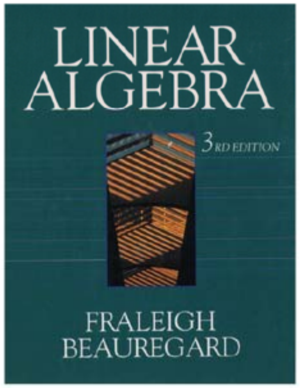
Linear Algebra

Chapter 8: Eigenvalues: Further Applications and Computations
Section 8.1. Diagonalization of Quadratic Forms—Proofs of Theorems



Page 417 Number 4(a)

Page 417 Number 4(a). Find the upper-triangular coefficient matrix of the quadratic form $x_1^2 - 2x_2^2 + x_3^2 + 6x_4^2 - 2x_1x_4 + 6x_2x_4 - 8x_1x_3$.

Solution. We take the coefficient of x_ix_j where $i \leq j$ as u_{ij}. All $u_{ij} = 0$ for $i > j$. So we have for the nonzero u_{ij}: $u_{11} = 1$, $u_{22} = -2$, $u_{33} = 1$, $u_{44} = 6$, $u_{14} = -2$, $u_{24} = 6$, $u_{13} = -8$. Hence

$$U = \begin{bmatrix} 1 & 0 & -8 & -2 \\ 0 & -2 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

□

Page 417 Number 4(b)

Page 417 Number 4(b). Find the symmetric coefficient matrix of the quadratic form $x_1^2 - 2x_2^2 + x_3^2 + 6x_4^2 - 2x_1x_4 + 6x_2x_4 - 8x_1x_3$.

Solution. From part (a) we have the upper-triangular coefficient matrix

$$U = \begin{bmatrix} 1 & 0 & -8 & -2 \\ 0 & -2 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

We take $a_{ij} = a_{ji} = u_{ij}/2$ for $i < j$ to get

$$A = \begin{bmatrix} 1 & 0 & -4 & -1 \\ 0 & -2 & 0 & 3 \\ -4 & 0 & 1 & 0 \\ -1 & 3 & 0 & 6 \end{bmatrix}.$$

□

Page 417 Number 12

Page 417 Number 12. Consider the quadratic form $x^2 + 2xy + y^2$. Find an orthogonal substitution that diagonalizes the quadratic form and find the diagonalized form.

Solution. For $x^2 + 2xy + y^2$, the upper-triangular coefficient matrix is $U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, so the symmetric coefficient matrix is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. We need the eigenvalues of A, so consider

$$\det(A - \lambda \mathcal{I}) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (1)^2$$

$$1 - 2\lambda + \lambda^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2),$$

so that the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 2$. Now for the eigenvectors.

Page 417 Number 12 (continued 1)

Solution (continued).

$\underline{\lambda_1 = 0}$ We consider the system

$$[A - \lambda_1 \mathcal{I} \mid \vec{0}] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \overset{R_2 \to R_2 - R_1}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So we need $\begin{array}{r} v_1 + v_2 = 0 \\ 0 = 0 \end{array}$ or $\begin{array}{l} v_1 = -v_2 \\ v_2 = v_2 \end{array}$ or with $r = v_2$ as a free variable, $\vec{v}_1 = r \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ where $r \in \mathbb{R}$, $r \neq 0$. We need a unit eigenvector so we take $r = 1/\sqrt{2}$ and $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

$\underline{\lambda_2 = 2}$ We consider the system

$$[A - \lambda_2 \mathcal{I} \mid \vec{0}] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \overset{R_1 \to R_1 + R_2}{\sim} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

Page 417 Number 12 (continued 2)

Solution (continued).

$$\overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So we need $\begin{array}{r} v_1 - v_2 = 0 \\ 0 = 0 \end{array}$ or $\begin{array}{l} v_1 = v_2 \\ v_2 = v_2 \end{array}$ or with $s = v_2$ as a free variable, $\vec{v}_2 = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $s \in \mathbb{R}$, $s \neq 0$. We need a unit eigenvector so we take $2 = 1/\sqrt{2}$ and $\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. So we consider the orthogonal matrix

$$C = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Page 417 Number 12 (continued 3)

Solution (continued). The orthogonal substitution is

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = C\vec{t} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} (-1/\sqrt{2})t_1 + (1/\sqrt{2})t_2 \\ (1/\sqrt{2})t_1 + (1/\sqrt{2})t_2 \end{bmatrix}.}$$

The diagonal form is then

$$\vec{x}^T A \vec{x} = \lambda_1 t_1^2 + \lambda_2 t_2^2 = 0t_1^2 + 2t_2^2 = \boxed{2t_2^2.}$$

□