Linear Algebra

Chapter 8: Eigenvalues: Further Applications and Computations

Section 8.2. Applications to Geometry—Proofs of Theorems

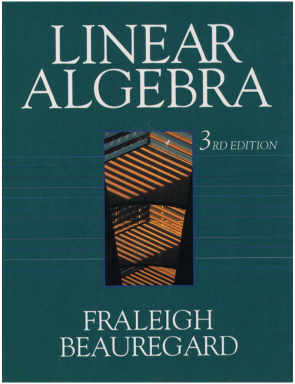

Table of contents

Theorem 8.2

Theorem 8.2. Classification of Second-Degree Plane Curves.
Every equation of the form

$$ax^2 + bxy + xy^2 + dx + ey + f = 0 \text{ for } a, b, c \text{ not all zero}$$

can be reduced to an equation of the form

$$\lambda_1 t_1^2 + \lambda_2 t^2 + gt_1 + ht_2 + k = 0$$

by means of an orthogonal substitution corresponding to a rotation of the plane. The coefficients λ_1 and λ_2 in the second equation are the eigenvalues of the symmetric coefficient matrix of the quadratic-form portion of the first equation. The curve describes a (possibly degenerate or empty)

$$\begin{array}{ll} \text{ellipse} & \text{if } \lambda_1\lambda_2 > 0 \\ \text{hyperbola} & \text{if } \lambda_1\lambda_2 < 0 \\ \text{parabola} & \text{if } \lambda_1\lambda_2 = 0. \end{array}$$

Theorem 8.2 (continued 1)

Proof. By Theorem 8.1, "Principal Axis Theorem," there is a 2×2 orthogonal matrix C with $\det(C) = 1$ such that $\begin{bmatrix} x \\ y \end{bmatrix} = C \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ and $ax^2 + bxy + cy^2 = \lambda_1 t_1^2 + \lambda_2 t_2^2$ where λ_1 and λ_2 are eigenvalues of the symmetric coefficient matrix of the quadratic form $ax^2 + bxy + xy^2$. With $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} c_{11}t_1 + c_{12}t_2 \\ c_{21}t_1 + c_{22}t_2 \end{bmatrix}$$

and so

$$dx + cy + f = fd(c_{11}t_1 + c_{12}t_2) + e(c_{21}t_1 + c_{22}t_2) + f$$
$$= (dc_{11} + ec_{21})t_1 + (dc_{12} + ec_{22})t_2 + f = gt_1 + ht_2 + k.$$

Theorem 8.2 (continued 1)

Proof. By Theorem 8.1, "Principal Axis Theorem," there is a 2×2 orthogonal matrix C with $\det(C) = 1$ such that $\begin{bmatrix} x \\ y \end{bmatrix} = C \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ and $ax^2 + bxy + cy^2 = \lambda_1 t_1^2 + \lambda_2 t_2^2$ where λ_1 and λ_2 are eigenvalues of the symmetric coefficient matrix of the quadratic form $ax^2 + bxy + xy^2$. With $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} c_{11}t_1 + c_{12}t_2 \\ c_{21}t_1 + c_{22}t_2 \end{bmatrix}$$

and so

$$dx + cy + f = fd(c_{11}t_1 + c_{12}t_2) + e(c_{21}t_1 + c_{22}t_2) + f$$

$$= (dc_{11} + ec_{21})t_1 + (dc_{12} + ec_{22})t_2 + f = gt_1 + ht_2 + k.$$

So the original equation can be reduced to the equation

$$\lambda_1 t_1^2 + \lambda_2 t_2^2 + gt_1 + ht_2 + k = 0.$$

Theorem 8.2 (continued 1)

Proof. By Theorem 8.1, "Principal Axis Theorem," there is a 2×2 orthogonal matrix C with $\det(C) = 1$ such that $\begin{bmatrix} x \\ y \end{bmatrix} = C \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$ and $ax^2 + bxy + cy^2 = \lambda_1 t_1^2 + \lambda_2 t_2^2$ where λ_1 and λ_2 are eigenvalues of the symmetric coefficient matrix of the quadratic form $ax^2 + bxy + xy^2$. With $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} c_{11}t_1 + c_{12}t_2 \\ c_{21}t_1 + c_{22}t_2 \end{bmatrix}$$

and so

$$dx + cy + f = fd(c_{11}t_1 + c_{12}t_2) + e(c_{21}t_1 + c_{22}t_2) + f$$

$$= (dc_{11} + ec_{21})t_1 + (dc_{12} + ec_{22})t_2 + f = gt_1 + ht_2 + k.$$

So the original equation can be reduced to the equation

$$\lambda_1 t_1^2 + \lambda_2 t_2^2 + gt_1 + ht_2 + k = 0.$$

Theorem 8.2 (continued 2)

Proof (continued). Since $\det(C) = 1$, the transformation $\vec{x} = C\vec{t}$ is a rotation (see page 413 of the text). So in the (t_1, t_2)-coordinate system, $\lambda_1 t_1^2 + \lambda_2 t_2^2 + gt_1 + ht_2 + k = 0$ is a conic section (possibly degenerate or empty) determined by the coefficients λ_1 and λ_2 as given in the statement of the theorem. □

Page 430 Number 16

Page 430 Number 16. Classify the quadric surface with equation $3x^2 + 2y^2 + 6xz + 3z^2 = 1$.

Solution. To find the symmetric coefficient matrix for the cross terms

$$3x^2 + 2y^2 + 6xz + 3z^2 = \frac{u_{11}}{2}x^2 + \frac{u_{22}}{2}y^2 + \frac{u_{33}}{2}x^2 + \frac{u_{12}}{2}xy + \frac{u_{21}}{2}yx + \frac{u_{13}}{2}xz + \frac{u_{31}}{2}zx + \frac{u_{23}}{2}yz + \frac{u_{32}}{2}zy,$$

we take $u_{11} = 6$, $u_{22} = 4$, $u_{33} = 6$, $u_{13} = u_{31} = 6$, and $u_{12} = u_{21} = u_{23} = u_{32} = 0$. Since $a_{ij} = a_{ji} = u_{ij}/2$ by Theorem 8.1.A, we have $a_{11} = 3$, $a_{22} = 2$, $a_{33} = 3$, $a_{13} = a_{31} = 3$, and $a_{12} = a_{21} = a_{23} = a_{32} = 0$.

Page 430 Number 16

Page 430 Number 16. Classify the quadric surface with equation $3x^2 + 2y^2 + 6xz + 3z^2 = 1$.

Solution. To find the symmetric coefficient matrix for the cross terms

$$3x^2 + 2y^2 + 6xz + 3z^2 = \frac{u_{11}}{2}x^2 + \frac{u_{22}}{2}y^2 + \frac{u_{33}}{2}x^2 + \frac{u_{12}}{2}xy$$

$$+\frac{u_{21}}{2}yx + \frac{u_{13}}{2}xz + \frac{u_{31}}{2}zx + \frac{u_{23}}{2}yz + \frac{u_{32}}{2}zy,$$

we take $u_{11} = 6$, $u_{22} = 4$, $u_{33} = 6$, $u_{13} = u_{31} = 6$, and $u_{12} = u_{21} = u_{23} = u_{32} = 0$. Since $a_{ij} = a_{ji} = u_{ij}/2$ by Theorem 8.1.A, we have $a_{11} = 3$, $a_{22} = 2$, $a_{33} = 3$, $a_{13} = a_{31} = 3$, and $a_{12} = a_{21} = a_{23} = a_{32} = 0$. So the symmetric coefficient matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{bmatrix}.$$

Page 430 Number 16

Page 430 Number 16. Classify the quadric surface with equation $3x^2 + 2y^2 + 6xz + 3z^2 = 1$.

Solution. To find the symmetric coefficient matrix for the cross terms

$$3x^2 + 2y^2 + 6xz + 3z^2 = \frac{u_{11}}{2}x^2 + \frac{u_{22}}{2}y^2 + \frac{u_{33}}{2}x^2 + \frac{u_{12}}{2}xy$$

$$+\frac{u_{21}}{2}yx + \frac{u_{13}}{2}xz + \frac{u_{31}}{2}zx + \frac{u_{23}}{2}yz + \frac{u_{32}}{2}zy,$$

we take $u_{11} = 6$, $u_{22} = 4$, $u_{33} = 6$, $u_{13} = u_{31} = 6$, and $u_{12} = u_{21} = u_{23} = u_{32} = 0$. Since $a_{ij} = a_{ji} = u_{ij}/2$ by Theorem 8.1.A, we have $a_{11} = 3$, $a_{22} = 2$, $a_{33} = 3$, $a_{13} = a_{31} = 3$, and $a_{12} = a_{21} = a_{23} = a_{32} = 0$. So the symmetric coefficient matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{bmatrix}.$$

Page 430 Number 16 (continued 1)

Page 430 Number 16. Classify the quadric surface with equation $3x^2 + 2y^2 + 6xz + 3z^2 = 1$.

Proof (continued). To use Theorem 8.3 (and the not following it) we only need the eigenvalues of A. We have

$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} 3-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & 3-\lambda \end{vmatrix} = -0 + (2-\lambda)\begin{vmatrix} 3-\lambda & 3 \\ 3 & 3-\lambda \end{vmatrix} - 0$$

$$(2-\lambda)((3-\lambda)^2 - 9) = (2-\lambda)(\lambda^2 - 6\lambda) = \lambda(2-\lambda)(\lambda - 6)$$

and so the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 2$, and $\lambda_3 = 6$. So, since one eigenvalue is zero and the other two are of the same sign, we know that the surface is either an elliptic paraboloid or an elliptic cylinder.

Page 430 Number 16 (continued 1)

Page 430 Number 16. Classify the quadric surface with equation $3x^2 + 2y^2 + 6xz + 3z^2 = 1$.

Proof (continued). To use Theorem 8.3 (and the not following it) we only need the eigenvalues of A. We have

$$\det(A-\lambda\mathcal{I}) = \begin{vmatrix} 3-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & 3-\lambda \end{vmatrix} = -0+(2-\lambda)\begin{vmatrix} 3-\lambda & 3 \\ 3 & 3-\lambda \end{vmatrix} - 0$$

$$(2-\lambda)((3-\lambda)^2 - 9) = (2-\lambda)(\lambda^2 - 6\lambda) = \lambda(2-\lambda)(\lambda-6)$$

and so the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 2$, and $\lambda_3 = 6$. So, since one eigenvalue is zero and the other two are of the same sign, we know that the surface is either an elliptic paraboloid or an elliptic cylinder.

Page 430 Number 16 (continued 2)

Page 430 Number 16. Classify the quadric surface with equation $3x^2 + 2y^2 + 6xz + 3z^2 = 1$.

Proof (continued). To further narrow the answer, we apply Step 3 of "Diagonalizing a Quadratic Form $f(\vec{x})$" from Section 8.1. We have that

$$3x^2 + 2y^2 + 6xz + 3y^2 = \lambda_1 t_1^2 + \lambda_2 t_2^2 + \lambda_3 t_3^2 - 2t_2^2 + 6t_3^2.$$

So in the (t_1, t_2, t_3)-coordinate system, the given equation becomes $2t_2^2 + 6t_3^2 = 1$, or $t_2^2/3 + t_3^2/1 = 1$, or

$$\frac{t_2^2}{(\sqrt{3})^2} + \frac{t_3^2}{(1)^2} = 1.$$

So the surface is in fact an elliptic cylinder (with semi-major axis $\sqrt{3}$ and semi-minor axis 1). $\square$

Page 430 Number 16 (continued 2)

Page 430 Number 16. Classify the quadric surface with equation $3x^2 + 2y^2 + 6xz + 3z^2 = 1$.

Proof (continued). To further narrow the answer, we apply Step 3 of "Diagonalizing a Quadratic Form $f(\vec{x})$" from Section 8.1. We have that

$$3x^2 + 2y^2 + 6xz + 3y^2 = \lambda_1 t_1^2 + \lambda_2 t_2^2 + \lambda_3 t_3^2 - 2t_2^2 + 6t_3^2.$$

So in the (t_1, t_2, t_3)-coordinate system, the given equation becomes $2t_2^2 + 6t_3^2 = 1$, or $t_2^2/3 + t_3^2/1 = 1$, or

$$\frac{t_2^2}{(\sqrt{3})^2} + \frac{t_3^2}{(1)^2} = 1.$$

So the surface is in fact an elliptic cylinder (with semi-major axis $\sqrt{3}$ and semi-minor axis 1). $\square$