SECTION 1.2
EXERCISE #43

1.2.43 For vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$, prove that $\vec{v}-\vec{w}$ and $\vec{v}+\vec{w}$ are perpendicular if and only if $\|\vec{v}\| = \|\vec{w}\|$.

Proof

Suppose $\vec{v}-\vec{w}$ and $\vec{v}+\vec{w}$ are perpendicular. Then by Definition 1.7, "Perpendicular or Orthogonal Vectors," $(\vec{v}-\vec{w})\cdot(\vec{v}+\vec{w}) = 0$. Now,

$(\vec{v}-\vec{w})\cdot(\vec{v}+\vec{w}) = (\vec{v}-\vec{w})\cdot\vec{v} + (\vec{v}-\vec{w})\cdot\vec{w}$ by the Distributive Law (Theorem 1.3)

$= \vec{v}\cdot(\vec{v}-\vec{w}) + \vec{w}\cdot(\vec{v}-\vec{w})$ by the Commutative Law (Theorem 1.3)

$= \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{v} - \vec{w}\cdot\vec{w}$ by the Distributive Law (Theorem 1.3)

$= \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{w} + \vec{v}\cdot\vec{w} - \vec{w}\cdot\vec{w}$ by the Commutative Law (Theorem 1.3)

$= \|\vec{v}\|^2 - \|\vec{w}\|^2$ since $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$.

So $(\vec{v}-\vec{w})\cdot(\vec{v}+\vec{w}) = \|\vec{v}\|^2 - \|\vec{w}\|^2 = 0$ and hence $\|\vec{v}\|^2 = \|\vec{w}\|^2$ and $\|\vec{v}\| = \|\vec{w}\|$.

Now suppose $\|\vec{v}\| = \|\vec{w}\|$. As shown above, $(\vec{v}-\vec{w})\cdot(\vec{v}+\vec{w}) = \|\vec{v}\|^2 - \|\vec{w}\|^2$ so $\|\vec{v}\| = \|\vec{w}\|$ implies $(\vec{v}-\vec{w})\cdot(\vec{v}+\vec{w}) = 0$. So by Definition 1.7, $\vec{v}-\vec{w}$ and $\vec{v}+\vec{w}$ are perpendicular. ∎