SECTION 1.4
NUMBER 23 (CORRECTED!)

2.4.23 Solve

$$\begin{aligned} x_1 \quad - 2x_3 + x_4 &= 6 \\ 2x_1 - x_2 + x_3 - 3x_4 &= 0 \\ 9x_1 - 3x_2 - x_3 - 7x_4 &= 4. \end{aligned}$$

Solution

Consider the augmented matrix

$$[A|\vec{b}] = \left[\begin{array}{cccc|c} \boxed{1} & 0 & -2 & 1 & 6 \\ 2 & -1 & 1 & -3 & 0 \\ 9 & -3 & -1 & -7 & 4 \end{array}\right] \xrightarrow[R_3 \to R_3 - 9R_1]{R_2 \to R_2 - 2R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 6 \\ 0 & -1 & 5 & -5 & -12 \\ 0 & -3 & 17 & -16 & -50 \end{array}\right] \xrightarrow{R_2 \to -R_2} \left[\begin{array}{cccc|c} \boxed{1} & 0 & -2 & 1 & 6 \\ 0 & \boxed{1} & -5 & 5 & 12 \\ 0 & \textcircled{-3} & 17 & -16 & -50 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 3R_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 6 \\ 0 & 1 & -5 & 5 & 12 \\ 0 & 0 & \boxed{2} & -1 & -14 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3/2} \left[\begin{array}{cccc|c} 1 & 0 & \textcircled{-2} & 1 & 6 \\ 0 & 1 & \textcircled{-5} & 5 & 12 \\ 0 & 0 & \boxed{1} & -\frac{1}{2} & -7 \end{array}\right] \xrightarrow[R_2 \to R_2 + 5R_3]{R_1 \to R_1 + 2R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -8 \\ 0 & 1 & 0 & \frac{5}{2} & -23 \\ 0 & 0 & 1 & -\frac{1}{2} & -7 \end{array}\right].$$

This corresponds to the system of equations:

$$\begin{aligned} x_1 \quad &= -8 \\ x_2 + \tfrac{5}{2} x_4 &= -23 \\ x_3 - \tfrac{1}{2} x_4 &= -7 \end{aligned}$$

or

$$\begin{aligned} x_1 &= -8 \\ x_2 &= -23 - (5/2)x_4 \\ x_3 &= -7 + (1/2)x_4 \\ x_4 &= \qquad x_4. \end{aligned}$$

SO

SECTION 2.4
NUMBER 23 (continued)

Let $r = x_4/2$ be a free variable. Then

$$x_1 = -8$$
$$x_2 = -23 - 5r \qquad \text{where } r \in \mathbb{R}.$$
$$x_3 = -7 + r$$
$$x_4 = 2r$$

In vector notation,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -8 \\ -23 \\ -7 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ -5 \\ 1 \\ 2 \end{bmatrix} \text{ where } r \in \mathbb{R}$$

is the general solution. □