SECTION 1.4
EXERCISE #25

1.4.25 Is $\vec{b} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}$ in the span of

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} -3 \\ -1 \\ 5 \end{bmatrix} ?$$

<u>Solution</u>

Well, the span of a set of vectors is the set of all linear combinations of the vectors, so we want $r_1, r_2, r_3 \in \mathbb{R}$ such that $r_1 \vec{v}_1 + r_2 \vec{v}_2 + r_3 \vec{v}_3 = \vec{b}$:

$$r_1 \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} + r_2 \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} + r_3 \begin{bmatrix} -3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \text{ or}$$

$$\begin{bmatrix} r_2 - 3r_3 \\ 2r_1 + 4r_2 - r_3 \\ 4r_1 - 2r_2 + 5r_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \text{ or } \begin{aligned} r_2 - 3r_3 &= 3 \\ 2r_1 + 4r_2 - r_3 &= 5 \\ 4r_1 - 2r_2 + 5r_3 &= 3 \end{aligned}$$

Consider the augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & 1 & -3 & 3 \\ 2 & 4 & -1 & 5 \\ 4 & -2 & 5 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} \boxed{2} & 4 & -1 & 5 \\ 0 & 1 & -3 & 3 \\ 4 & -2 & 5 & 3 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_1}$$

$$\left[\begin{array}{ccc|c} 2 & 4 & -1 & 5 \\ 0 & \boxed{1} & -3 & 3 \\ 0 & -10 & 7 & -7 \end{array}\right] \xrightarrow{R_3 \to R_3 + 10R_2} \left[\begin{array}{ccc|c} \boxed{2} & 4 & -1 & 5 \\ 0 & \boxed{1} & -3 & 3 \\ 0 & 0 & \boxed{-23} & 23 \end{array}\right].$$

Since the 3x3 coefficient matrix has pivots in (all) 3 columns then there is a solution r_1, r_2, r_3 (by backsubstitution) and so $\boxed{\text{YES}}$ $\vec{b}$ is in the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3$. □