SECTION 1.5

NUMBER 15

1.5.15 Find three linear equations that express x, y, z in terms of r, s, t if

$$
\begin{aligned}
2x + y + 4z &= r \\
3x + 2y + 5z &= s \\
-y + z &= t.
\end{aligned}
$$

Solution

Let's represent this as $A\vec{x} = \vec{b}$ and find A^{-1}. Then $\vec{x} = A^{-1}\vec{b}$. Consider

$$
[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 3 & 2 & 5 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]
$$

$$
\xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2
$$

$$
\left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & -1 & 1 & 0 \\ 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 3 & -2 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]
$$

$$
\xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & -1 & 1 & 0 \\ 0 & \boxed{1} & -2 & -3 & 2 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_2]{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & -2 & -3 & 2 & 0 \\ 0 & 0 & -1 & -3 & 2 & 1 \end{array}\right]
$$

$$
\xrightarrow{R_3 \to -R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 2 & -1 & 0 \\ 0 & 1 & -2 & -3 & 2 & 0 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right] \xrightarrow[R_2 \to R_2 + 2R_3]{R_1 \to R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 5 & 3 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]
$$

So

$$
A^{-1} = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}.
$$

SECTION 1.5
NUMBER 15 (continued)

So $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\vec{b} = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$

$= \begin{bmatrix} -7r + 5s + 3t \\ 3r - 2s - 2t \\ 3r - 2s - t \end{bmatrix}$ and

$$\boxed{\begin{aligned} x &= -7r + 5s + 3t \\ y &= 3r - 2s - 2t \\ z &= 3r - 2s - t \end{aligned}}$$

$\square$