SECTION 1.5
EXERCISE #25

1.5.25 (a) If A is invertible, is $A + A^T$ necessarily invertible?

Solution

Certainly a matrix of all 0's is not invertible (by Theorem 1.12, "Conditions for A^{-1} to Exist," say). So we try to find invertible (square) matrix A such that $A + A^T$ is a matrix of all 0's. This can be accomplished for 2×2 matrices with $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then $A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Both A and A^T are invertible since their columns span $\mathbb{R}^2$ (the columns form the standard basis for $\mathbb{R}^2$ where each standard basis vector has been multiplied by ± 1); see Theorem 1.12. Also, $A + A^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so $A + A^T$ is not invertible.

For a 4×4 example, we can take $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$ and this easily generalizes to $n \times n$ matrices for n even. So to answer the stated question $\boxed{\text{NO!}}$

(b) If A is invertible, is $A + A$ invertible?

Solution

Since A is invertible then A^{-1} exists. Now

$(A+A)(\frac{1}{2}A^{-1}) = A(\frac{1}{2}A^{-1}) + A(\frac{1}{2}A^{-1})$ by the Distributive Law of Matrix Multiplication (Theorem 2.3.A)

$= \frac{1}{2}(AA^{-1}) + \frac{1}{2}(AA^{-1})$ by Scalars "Pull Through" (Theorem 2.3.A)

$= \frac{1}{2}I + \frac{1}{2}I = 1I = I.$

By Theorem 1.11, "A Commutative Property," $(\frac{1}{2}A^{-1})(A+A) = I$ and so $A+A$ is invertible with inverse $\frac{1}{2}A^{-1}$! $\square$