SECTION 1.5
EXERCISE #27

1.5.27 Let A and B be $n \times n$ matrices with A invertible.

(a) Prove that $AX = B$ has the unique solution $X = A^{-1}B$.

Proof

First, with $X = A^{-1}B$ we have

$AX = A(A^{-1}B) = (AA^{-1})B$ by Associativity of Matrix Multiplication (Theorem 1.3.A)

$= \mathcal{I} B = B$

and so $X = A^{-1}B$ actually is a solution.

Now suppose X' is another solution to $AX = B$; that is, suppose $AX' = B$. Then

$AX' - AX = B - B = 0$ (where 0 denotes the $n \times n$ matrix of all 0's),

and $A(X' - X) = 0$ by Distributive Laws of Matrix Multiplication (Theorem 1.3.A). So

$A^{-1}(A(X' - X)) = A^{-1}0$ or $(A^{-1}A)(X' - X) = 0$ (by Associativity of Matrix Multiplication) and so

$\mathcal{I}(X' - X) = 0$, or $X' - X = 0$ or $X' = X + 0 = X$.

So any "other" solution X' actually equals X and X is the unique solution. ■

(b) Prove that $X = A^{-1}B$ can be found by the following row reduction: $[A \mid B] \sim [\mathcal{I} \mid X]$. That is, if the matrix A is reduced to the identity matrix $\mathcal{I}$, then the matrix B will be reduced to $A^{-1}B$.

Proof

[This result shows an importance of elementary matrices!]

Let $E_1, E_2, \ldots, E_k$ be the $n \times n$ elementary matrices that reduce $[A \mid B]$ to $[\mathcal{I} \mid X]$; that is,

$$E_k E_{k-1} \cdots E_2 E_1 [A \mid B] = [\mathcal{I} \mid X].$$

SECTION 1.5
EXERCISE #27 (continued)

Applying the row operations corresponding to elementary matrices $E_1, E_2, \ldots, E_k$ to A and B individually we have

$$E_k E_{k-1} \cdots E_2 E_1 A = I \text{ and } E_k E_{k-1} \cdots E_2 E_1 B = X.$$

By Theorem 1.11, "A Commutative Property,"

$$A(E_k E_{k-1} \cdots E_2 E_1) = I \text{ and so } A^{-1} = E_k E_{k-1} \cdots E_2 E_1.$$

So $X = E_k E_{k-1} \cdots E_2 E_1 B = A^{-1} B$, as claimed. ∎