SECTION 1.5
EXERCISE #29

1.5.29 An $n \times n$ matrix is nillpotent if $A^r = 0$ (the $n \times n$ zero matrix) for some positive integer n.

(a) Give an example of a nonzero nillpotent 2×2 matrix.

Solution

We can take either $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ since squaring these matrices gives the 2×2 zero matrix. We can replace the 1's with any nonzero number for more examples. □

(b) Prove that if A is an invertible $n \times n$ matrix then A is not nillpotent.

Proof

ASSUME A is both invertible and nillpotent; say $A^r = 0$. Then

$$0 = 0(A^{-1})^{r-1} = A^r (A^{-1})^{r-1}$$

$$= \underbrace{AA\cdots AA}_{r \text{ times}} \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{r-1 \text{ times}} = \underbrace{AA\cdots A}_{r-1 \text{ times}} \mathcal{I} \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{r-2 \text{ times}}$$

$$= \underbrace{AA\cdots A}_{r-2 \text{ times}} \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{r-2 \text{ times}} = \cdots = A\mathcal{I} = A.$$

That is, $A = 0$. But the zero matrix is not invertible (by Theorem 2.12, "Conditions for A^{-1} to Exist," say), a CONTRADICTION! So the assumption that A is both invertible and nillpotent is false. That is, if A is invertible then it is not nillpotent. ■

Note This is a standard proof by contradiction. We assume the negation of the result and show that this implies a contradiction.