

SECTION 2.5  
EXERCISE #31

2.5.31 Show that  $A = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  is nilpotent

(that is,  $A^r = O$ , the  $4 \times 4$  zero matrix, for some positive integer  $r$ ).

Solution

We start taking powers:

$$A^2 = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & a_1 b_1 & a_1 b_2 + a_2 c_1 \\ 0 & 0 & 0 & b_1 c_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^4 = (A^2)^2 = \begin{bmatrix} 0 & 0 & a_1 b_1 & a_1 b_2 + a_2 c_1 \\ 0 & 0 & 0 & b_1 c_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & a_1 b_1 & a_1 b_2 + a_2 c_1 \\ 0 & 0 & 0 & b_1 c_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = O \quad \text{and so } A \text{ is nilpotent} \\ \text{and we can take } r=4. \quad \square$$