SECTION 2.5

EXERCISE #37

2.5.37 Let A be a square matrix and let E be an elementary matrix of the same size. Find the effect on A of multiplying A on the <u>right</u> by E.

<u>Solution</u>

Let's first experiment. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and we choose one of each of the types of elementary matrices:

- Multiply row 2 by k: $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$,
- Interchange row 1 and row 2: $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,
- Add k times row 1 to row 3: $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$.

Multiplying on the right by the elementary matrices gives:

$$AE_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2k & 3 \\ 4 & 5k & 6 \\ 7 & 8k & 9 \end{bmatrix}$$

$$AE_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{bmatrix}$$

$$AE_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+3k & 2 & 3 \\ 4+6k & 5 & 6 \\ 7+9k & 8 & 9 \end{bmatrix}$$

So the elementary matrix multiplication is producing <u>column</u> operations (multiplying column 2 by k, swapping columns 1 and 2, and adding k times column 3 to column 1, respectively). Elementary matrices E_1 and E_2 are symmetric and the row operations correspond directly to the same column operations. However, E_3 is not symmetric and it represents adding k times row 1 to row 3 OR adding k times column 3 to column 1 (notice how the 1 and 3 interchange roles here). But we are justified in stating:

> If E is an elementary matrix which results by applying one column operation to the identity matrix then multiplication of A on the right by E performs the same column operation on A.

We can justify this more quantitatively by letting E be an elementary matrix representing one elementary column operation and observing that E^T represents the same elementary operation, but applied to rows instead of columns (for example, E_3^T represents adding k times row 3 to row 1 — the same column operations as represented by E_3). Then
$(AE)^T = E^T A^T$ by Note 1.3.B. So E^T affects the rows of A^T (which are the columns of A) the same way E affects the columns of A. $\square$