SECTION 1.6

NUMBER 31

1.6.31 Find a basis for the nullspace of the matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 1 \\ 1 & 7 & 2 \\ 6 & -2 & 0 \end{bmatrix}.$$

Solution

The nullspace of A is, by definition, the set of all solutions to the system of equations $A\vec{x} = \vec{0}$. So we consider the augmented matrix $[A|\vec{0}]$, row reduce it to reduced row echelon form, and introduce free variables. Each vector associated with a free variable will be a basis vector for the nullspace of A. So:

$$[A|\vec{0}] = \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 5 & 2 & 1 & 0 \\ 1 & 7 & 2 & 0 \\ 6 & -2 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} \boxed{1} & 7 & 2 & 0 \\ 5 & 2 & 1 & 0 \\ 2 & 3 & 1 & 0 \\ 6 & -2 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 6R_1}]{R_2 \to R_2 - 5R_1} \left[\begin{array}{ccc|c} 1 & 7 & 2 & 0 \\ 0 & -33 & -9 & 0 \\ 0 & -11 & -3 & 0 \\ 0 & -44 & -12 & 0 \end{array}\right] \xrightarrow[R_4 \to R_4/(-4)]{R_2 \to R_2/(-3)} \left[\begin{array}{ccc|c} 1 & 7 & 2 & 0 \\ 0 & 11 & 3 & 0 \\ 0 & -11 & -3 & 0 \\ 0 & 11 & 3 & 0 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 7 & 2 & 0 \\ 0 & \boxed{11} & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2/(11)} \left[\begin{array}{ccc|c} 1 & 7 & 2 & 0 \\ 0 & \boxed{1} & 3/11 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

SECTION 1.6
NUMBER 31 (continued)

$$R_1 \to R_1 - 7R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1/11 & 0 \\ 0 & 1 & 3/11 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$ ← RREF ☺

If $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then the associated system

of equations is $\begin{aligned} x \quad + \tfrac{1}{11} z &= 0 \\ y + \tfrac{3}{11} z &= 0 \end{aligned}$ or $\begin{aligned} x &= \tfrac{-1}{11} z \\ y &= \tfrac{-3}{11} z. \\ z &= z \end{aligned}$

① Let $r = z$ be a free variable. Then the nullspace is $\{\vec{x} \in \mathbb{R}^3 \mid [x, y, z]\}$

$= \{\vec{x} \in \mathbb{R}^3 \mid [x, y, z] = [\tfrac{-1}{11} r, \tfrac{-3}{11} r, r], r \in \mathbb{R}\}.$

$= \{\vec{x} \in \mathbb{R}^3 \mid \tfrac{r}{11}[-1, -3, 11], r \in \mathbb{R}\}.$

$\boxed{\text{SO, a basis for nullspace is } \{\tfrac{1}{11}[-1, -3, 11]\}.}$

② Alternate solution: Let $r = \frac{1}{11} z$ be a free variable. Then we get:

$\boxed{\{[-1, -3, 11]\} \text{ is a basis for the nullspace.}}$

□