SECTION 1.6
EXERCISE #33

1.6.33 Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ and $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ be vectors in a vector space V. Give a necessary and sufficient condition involving linear combinations for $sp(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) = sp(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$.

<u>Solution</u>

If $sp(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) = sp(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$ then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in sp(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$ and so each $\vec{v}_i$ must be a linear combination of $\vec{w}_1, \vec{w}_2, \ldots, w_k$ (for $i = 1, 2, \ldots, k$). Similarly, $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m \in sp(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$ and each $\vec{w}_i$ must be a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ (for $i = 1, 2, \ldots, m$). So this condition is necessary for equality of the spans. We now show that it is also sufficient for equality.

Suppose each $\vec{v}_i$ is a linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ and each $\vec{w}_i$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$. Let $\vec{v} \in sp(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$. Then $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$. Since each $\vec{v}_i$ is in $sp(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$ then we have

$$\vec{v}_i = b_{i1}\vec{w}_1 + b_{i2}\vec{w}_2 + \cdots + b_{im}\vec{w}_m \quad \text{for } i = 1, 2, \ldots, k.$$

$$\begin{aligned} \text{So } \vec{v} &= a_1(b_{11}\vec{w}_1 + b_{12}\vec{w}_2 + \cdots + b_{1m}\vec{w}_m) \\ &\quad + a_2(b_{21}\vec{w}_1 + b_{22}\vec{w}_2 + \cdots + b_{2m}\vec{w}_m) \\ &\quad + \cdots + a_k(b_{k1}\vec{w}_1 + b_{k2}\vec{w}_2 + \cdots + b_{km}\vec{w}_m) \\ &= (a_1 b_{11} + a_2 b_{21} + \cdots + a_k b_{k1})\vec{w}_1 \\ &\quad + (a_1 b_{12} + a_2 b_{22} + \cdots + a_k b_{k2})\vec{w}_2 \\ &\quad + \cdots + (a_1 b_{k1} + a_2 b_{k2} + \cdots + a_k b_{km})\vec{w}_m \end{aligned}$$

and hence $\vec{v} \in sp(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$. Since $\vec{v}$ is an arbitrary element of $sp(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$ then $sp(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) \subset sp(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$. That is, every vector in $sp(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$

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EXERCISE #33 (continued)

is also in $sp(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$. Similarly, for $\vec{w} \in sp(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$ we have $\vec{w} = c_1 \vec{w}_1 + c_2 \vec{w}_2 + \cdots + c_m \vec{w}_m$ and since each $\vec{w}_i$ is in $sp(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$ then we have

$\vec{w}_i = d_{i1} \vec{v}_1 + d_{i2} \vec{v}_2 + \cdots + d_{ik} \vec{v}_k$ for $i = 1, 2, \ldots, m$.

So

$$\begin{aligned} \vec{w} &= c_1 (d_{11} \vec{v}_1 + d_{12} \vec{v}_2 + \cdots + d_{1k} \vec{v}_k) \\ &\quad + c_2 (d_{21} \vec{v}_1 + d_{22} \vec{v}_2 + \cdots + d_{2k} \vec{v}_k) \\ &\quad + \cdots + c_m (d_{m1} \vec{v}_1 + d_{m2} \vec{v}_2 + \cdots + d_{mk} \vec{v}_k) \\ &= (c_1 d_{11} + c_2 d_{21} + \cdots + c_m d_{m1}) \vec{v}_1 \\ &\quad + (c_1 d_{12} + c_2 d_{22} + \cdots + c_m d_{m2}) \vec{v}_2 \\ &\quad + \cdots + (c_1 d_{1k} + c_2 d_{2k} + \cdots + c_m d_{mk}) \vec{v}_k \end{aligned}$$

and hence $\vec{w} \in sp(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$. As above, this shows $sp(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m) \subset sp(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$. Therefore the linear combination condition implies $sp(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) = sp(\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m)$.

So the linear combination condition is necessary and sufficient for the equality of the spans. ∎