SECTION 2.1
EXERCISE #35

2.1.35 Give an example of a 3×3 nonzero singular matrix A and two independent column vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ such that $A\vec{v}$ and $A\vec{w}$ are linearly independent. Also give an example of a 3×3 singular matrix A_1 and two independent vectors $\vec{v}_1$ and $\vec{w}_1$ such that $A_1\vec{v}_1$ and $A_1\vec{w}_1$ are still linearly independent.

Solution

We search for a matrix A and vector $\vec{v}$ such that $A\vec{v} = \vec{0}$. Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Also let $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then A is singular (by Theorem 1.12, "Conditions for A^{-1} to Exist") and

$$A\vec{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad A\vec{w} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

So $1\,A\vec{v} + 0\,A\vec{w} = \vec{0}$ and this is a dependence relation so $A\vec{v}$ and $A\vec{w}$ are linearly dependent.

With $A_1 = A$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\vec{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,

we have that A_1 is singular but

$$A_1\vec{v}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A\vec{w}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and $r_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$ and so $r_1 = r_2 = 0$.

Therefore $A\vec{v}_1$ and $A\vec{w}_1$ are linearly independent, as desired. $\square$